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PRECISE ANALYSIS OF AN OPTIMAL PERTURBATION ESTIMATION AND CONT--ETC(U)

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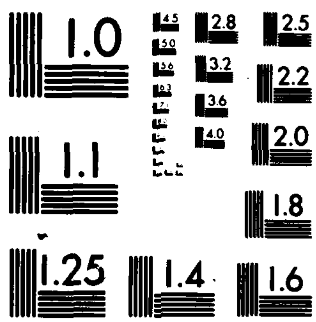
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Precise Analysis of an Optimal Perturbation Estimation and Control Problem

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A mathematically rigorous analysis is performed of a scalar discrete-time stochastic optimal control problem. This problem differs from the standard linear-quadratic-Gaussian one only by the presence of a quadratic term in the state and process noise with a small coefficient. A certain formal approximation to the optimal control is shown to converge asymptotically in probability to an optimal control with respect to this coefficient.

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CONTENTS

PRECISE ANALYSIS OF AN OPTIMAL PERTURBATION ESTIMATION AND CONTROL PROBLEM

1. INTRODUCTION

There is considerable interest in state-estimation problems in which a state variable evolves according to linear dynamics with additive Gaussian white noise and in which linear state measurements are available which are also corrupted by additive Gaussian white noise. There is also interest in associated control problems in which the control enters the dynamics linearly and in which the objective is to find a feedback control law, which specifies the current control as a function of the currently available state measurements, that minimizes the prior expected value of a quadratic performance criterion. These cases are important partly because they are the forms resulting from first-order descriptions of noise-induced perturbations from nominal behavior in a wider class of state-estimation and optimal-control problems.

If the description of these perturbations is carried out to one higher order of accuracy, the effect is typically to introduce quadratic terms in the dynamics and state measurements and cubic terms in the performance criterion of the control problem. The resulting estimation and control problems can often be rescaled so that the state, control, and measurement perturbations are of order unity, and the coefficients of the added higher-degree terms become the relatively small quantities.

A formal analysis has been made [1] of a class of estimation and optimal-control problems with this latter type of structure. The results had the formal appearance of giving approximations to the conditional probability density of the current state, given the currently available measurements, for the state estimation problem, and to the control generated by an optimal control law, in the control problem, which were accurate to first order in the small coefficients. Similar formal results were obtained in Ref. 2 for the corresponding "smoothing problem," i.e., an approximation to the conditional density of the current state given future as well as past measurements. Approximations of this sort are of interest because they show how the solutions start to change as the problems begin to depart from the familiar linear-quadratic-Gaussian form. And if such an estimation or control problem arises from the sort of higher-order perturbation analysis described above, this degree of accuracy in the solution is all that is formally consistent with that of the problem formulation anyway. Also, the first-order approximation to the conditional state probability density in all cases has the interesting property of being (at least in formal appearance) the first-order Edgeworth expansion of that density. Questions naturally arise about the validity of this formalism, however, namely as to the exact sense of the first-order accuracy and the range of conditions under which this accuracy holds.

Some mathematically precise answers to such questions are developed in this report for the simplest nontrivial case of a scalar discrete-time problem in which the only higher-degree term present is a product of the state and noise variables in the dynamics (state-dependent process noise). In particular, limits are established on the error in the first-order formal approximation to the conditional state density, for all but a set of realizations of negligible prior probability, which are sufficiently strong to guarantee that the corresponding formal first-order approximation to the optimal control is indeed accurate to first order in the perturbation parameters. Furthermore, the number of epochs in the estimation and control problems can go to infinity as the perturbation parameters approach zero, while the validity

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of these results is maintained. In order to establish the accuracy of the optimal control approximation, it was found necessary to establish limits on both the size and the fluctuation of the error in the conditional state density approximation for the corresponding estimation problem, the latter limit being in the form of modified Lipschitz conditions.

Because of their limited scope, these results are more exploratory than definitive. The hope is that they can serve as a guide to what sort of results could be achieved and what some of the phenomena are that might be encountered in a more general investigation. Even in the limited context here, rather elaborate constructions seemed to be required in the analysis, and the proofs become highly computational. One difficulty, for example, is that the conditional state probability density actually diverges for very large values of its argument. Presumably, some corresponding pathology would also arise in the continuous-time version of this state-estimation problem. Thus it might be useful to devote some effort to developing more sophisticated and elegant concepts and methods for dealing with the phenomena encountered here before proceeding with this sort of mathematically precise analysis in more general contexts.

2. THE STATE-ESTIMATION PROBLEM

We consider a discrete-time state estimation problem in which a real, scalar state variable x evolves according to the transition equation

$$x_{i+1} = f_i x_i + u_i + (1 + \psi_i x_i) w_i; \quad i = 0, \ldots, N-1, \tag{1}$$

where f_i, u_i, and ψ_i are known parameters and the w_i are independent zero-mean normal random variables with variance var $(w_i) = q_i$. At each epoch *except* $i = 0$, a noisy measurement of the current state is received, and

$$z_i = x_i + n_i; \quad i = 1 \ldots, N; \tag{2}$$

where the n_i are independent zero-mean normal random variables, independent of the w_i, such that var $(n_i) = r_i$. The initial value x_0 of the state is, a priori, a normal random variable, independent of the w_i and n_i, with mean $\bar{x}_0$ and variance v_0. It is assumed that there are positive constants F, Q, and R such that

$$\begin{aligned} 1 > |f_i| &\geq F; \quad i = 0, \ldots, N-1; \\ |q_i| &\leq QF^2; \quad i = 0, \ldots, N-1; \\ R &\leq v_0 \leq Q; \end{aligned}$$

and

$$R \leq |r_i| \leq Q; \quad i = 1, \ldots, N.$$

The quantity h, defined as

$$h = \max \{|\psi_i|: i = 0, \ldots, N-1\},$$

is treated as a perturbation variable in the ensuing analysis. In other words, for fixed values of the other parameters in the system of Eqs. (1) and (2), results are obtained as functions of h for all ψ_i sequences with this maximum magnitude, h generally being considered a relatively small quantity. A further restriction is assumed for the parameters of the unperturbed problem; this is described in a later section. For convenience, we also denote the sequences

$$Z_i = \{z_1, \ldots, z_i\}; \quad i = 1, \ldots, N$$

and

$$Y_i = \{u_0, \ldots, u_i\}; \quad i = 0, \ldots, N-1,$$

with Z_0 denoting the empty sequence.

The basic objective here is to establish error bounds for certain approximations of the conditional probability density functions of the state

$$p(x_{i+1}/Z_i); \quad i = 0, \ldots, N-1$$

and

$$p(x_i/Z_i); \quad i = 1, \ldots, N.$$

In the unperturbed problem corresponding to Eqs. (1) and (2), $h = \psi_i = 0$; $i = 0, \ldots, N-1$; and it is well known that these conditional state densities are all normal. Their means and variances are given by a simple case of the standard Kalman-Bucy filter, the variance equations for which are listed here for later reference:

$$\mu_{i+1} = f_i^2 p_i + q_i; \quad i = 0, \ldots, N-1 \quad (\text{variance of } p(x_{i+1}/Z_i)); \tag{3}$$

$$p_i = \frac{\mu_i r_i}{\mu_i + r_i}; \quad i = 1, \ldots, N; \quad p_0 = v_0 \quad (\text{variance of } p(x_i/Z_i)). \tag{4}$$

For nonzero values of h, however, these densities do not even exist in general for large values of their arguments. For example, it is shown in Appendix A that if ψ_0 and q_0 are nonzero, then $p_{x_1}(1/\psi_0)$ does not exist, in the sense that

$$\lim_{\epsilon \to 0} \left[\frac{1}{2\epsilon} \Pr\left\{ \frac{1}{\psi_0} - \epsilon < x_1 \leqslant \frac{1}{\psi_0} + \epsilon \right\} \right] = \infty.$$

Hence it is only reasonable to seek approximations to such probability densities within wide but bounded ranges, which leads to the somewhat elaborate approximation concepts considered here.

3. ESTIMATION EQUATION SYSTEM I FROM FIRST-ORDER FORMAL ANALYSIS

A formal analysis similar to that described in Ref. 1 suggests that the above conditional state densities can be approximated to first order in h, in some reasonable sense, as those corresponding to

$$p_{\frac{x_{i+1} - \mathring{x}_{i+1}}{\sqrt{m_{i+1}}}/Z_i}(t/Z_i) \cong \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} \left[1 + \frac{1}{3} \bar{\eta}_{i+1} (t^3 - 3t)\right] \tag{5}$$

and

$$p_{\frac{x_i - \hat{x}_i}{\sqrt{v_i}}/Z_i}(t/Z_i) \cong \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} \left[1 + \frac{1}{3} \eta_i (t^3 - 3t)\right], \tag{6}$$

according to the well-known formula for the transformation of probability densities, where x^*_{i+1}, $\tilde{x}_i$, m_{i+1}, v_i, $\bar{\eta}_{i+1}$, and η_i are the functions of Z_i specified by the following equations, called Equation System I for future reference:

$$x^*_{i+1} = f_i \tilde{x}_i + u_i; \tag{7}$$

$$m_{i+1} = f_i^2 v_i + q_i (1 + \psi_i \tilde{x}_i)^2; \tag{8}$$

$$\eta^*_{i+1} = \left[\frac{f_i^2 v_i}{m_{i+1}}\right]^{3/2} \left[\eta_i + \frac{3 q_i \psi_i (1 + \psi_i \tilde{x}_i)}{f_i^2 \sqrt{v_i}}\right]; \tag{9}$$

$$\tilde{x}_i = x_i^* + \frac{m_i}{m_i + r_i}(z_i - x_i^*) + \left[\frac{\sqrt{m_i}}{m_i + r_i}\right]^3 r_i \eta_i^* [(z_i - x_i^*)^2 - (m_i + r_i)]; \;\; \tilde{x}_0 \text{ as given;} \tag{10}$$

$$v_i = \frac{m_i r_i}{m_i + r_i}\left[1 + \frac{\sqrt{m_i}\, r_i \eta_i^* (z_i - x_i^*)}{(m_i + r_i)^2}\right]^2; \;\; v_0 \text{ as given;} \tag{11}$$

and

$$\eta_i = \left[\frac{r_i}{m_i + r_i}\right]^{3/2} \eta_i^*; \;\; \eta_0 = 0. \tag{12}$$

3.1 Further Restrictions Assumed for the Unperturbed Problem

In terms of the unperturbed problem parameters, let

$$b = k \max_i \{\mu_i, p_i, 1\}$$

and

$$a = \frac{1}{k} \min_i \{\mu_i, p_i\}$$

for some constant $k > 1$, and let $G_i(h, k)$ denote the set of sequence pairs (Z_i, Y_{i-1}) for which, for all $\{\psi_0, \ldots, \psi_{N-1}\}$ with $\max \{|\psi_i|\} = h$,

Condition 1: $|\eta_j|, |\eta_j^*| < 3h\sqrt{b}$;

Condition 2: $|v_j|, |m_j| \in [a, b]$;

Condition 3: $|x_j^*|, |\tilde{x}_j| < \ln (1/h)$;

and

Condition 4: $\dfrac{(z_j - x_j^*)^2}{m_j + r_j} < 8 \ln (1/h)$

for $j = 1, \ldots, i$ and $i = 1, \ldots, N$. It is further assumed that the unperturbed problem parameters $\tilde{x}_0$, v_0, $\{f_j, q_j, r_{j+1};\ j = 0, \ldots, N-1\}$ are such that there exist a $k > 1$ giving a and b as above and positive numbers $\bar{h}$ and c, such that for every $h < \bar{h}$ and every $i = 1, \ldots, N$, $(Z_i, Y_{i-1}) \in G_i (h, k)$ if

$$\max\{|x_0 - \tilde{x}_0|, |n_i|, |w_0|, \max_{j=1,\ldots,i-1} \{|w_j|, |n_j|\}\} < c\sqrt{\ln (1/h)}$$

and

$$\max\{|u_j|: j = 0, \ldots, i-1\} < c \ln (1/h).$$

In this case, it follows easily from the independence of x_0, the w_j, and the n_j and from the inequality

$$\Pr\{|x| < t\} < \frac{e^{-\frac{t^2}{2}}}{t\sqrt{2\pi}} \; ; \; t > 0; \tag{13}$$

for a standardized normal random variable x that, a priori,

$$\Pr\{(Z_i, Y_{i-1}) \in G_i : i = 1, \ldots, N\} \geqslant 1 - \frac{N\sigma^2 h^{\frac{c^2}{2\sigma^2}}}{c\sqrt{2\pi}\ \ln(1/h)},$$

where

$$\sigma^2 = \max\{v_0,\ q_0,\ \max_{i=1,\ldots,N}\{q_i,\ r_i\}\},$$

if $|u_i| < c\ \ln(1/h)$ for $i = 0, \ldots, N-1$. This bound is significant because, even if the number of epochs N grows in perturbed problems with decreasing h as

$$N = \sqrt{\ln(1/h)},$$

so that $\lim_{h \to 0} N = \infty$, this prior probability, and hence also that of $(Z_i, Y_{i-1}) \in G_i$ for $i = 1, \ldots, N$, approaches unity in the limit as $h \to 0$.

As a verification that the ensuing analysis does not take place in a vacuum, it is shown in Appendix B that the preceding assumption is indeed valid for the specific case in which

$$\begin{aligned} &\tilde{x}_0 = 0, \\ &v_0 = 1, \\ &\left.\begin{aligned} &f_i = 1/2 \\ &q_i = r_{i+1} = 1 \end{aligned}\right\} i = 0, \ldots, N-1, \end{aligned}$$

with $k = 12$, $\bar{h} = e^{-5}$, and $c = 0.02$.

3.2 Lipschitz Conditions

For future use it is helpful to define

$$\bar{\lambda}_i = \eta_i\, v_i^{3/2} \; ; \; i = 1, \ldots, N \tag{14}$$

and

$$\lambda_i^* = \eta_i^*\, m_i^{3/2} \; ; \; i = 0, \ldots, N; \tag{15}$$

which are first-order approximations for one half the third central moments of the conditional probability distributions of x_i if, respectively, Z_{i-1} and Z_i are given. Equation System I can also be expressed in terms of these variables if we use Eq. (15) to replace η_i^* by λ_i^* in Eqs. (10) and (11) and replace Eqs. (9) and (12) by

$$\lambda_{i+1}^* = f_i^3\, \bar{\lambda} + 3 f_i\, v_i\, q_i\, \psi_i (1 + \psi_i\, \tilde{x}_i) \tag{16}$$

and

$$\bar{\lambda}_i = \left[\frac{r_i}{m_i + r_i}\right]^3 \left[1 + \frac{r_i \lambda_i^* (z_i - x_i^*)}{m_i (m_i + r_i)^2}\right]^3 \lambda_i^* ; \; \bar{\lambda}_0 = 0. \tag{17}$$

Lemma 1: In the context of the previous sections there exists an $h^* > 0$ such that if $h \leqslant h^*$ and if, for any $i = 0, \ldots, N$, (Z_i, Y_{i-1}) and $(Z_i, Y_{i-1})' \in G_i$ differ in at most one component, whose values are denoted respectively by ρ and ρ', then

$$|\tilde{x}_i [(Z_i, Y_{i-1})] - \tilde{x}_i[(Z_i, Y_{i-1})']| \leqslant \Gamma_i \, |\rho - \rho'|, \tag{18}$$

$$|V_i[(Z_i, Y_{i-1})] - v_i[(Z_i, Y_{i-1})']| \leqslant \Gamma_i \, h |\rho - \rho'|, \tag{19}$$

and

$$|\bar{\lambda}_i [(Z_i, Y_{i-1})] - \bar{\lambda}_i[(Z_i, Y_{i-1})']| \leqslant \Gamma_i h^2 |\rho - \rho'|, \tag{20}$$

where Γ_i is defined by the recursion

$$\Gamma_{k+1} = 1 + \frac{4b^2}{R} + \left[\left[2 + \frac{2}{R}\right](1 + 3Q) + \frac{6b^2}{R} + \frac{4b}{R}\left[2 + \frac{3b}{R}\right] + \frac{1}{R^2}\left[1 + 3Q + \frac{6b}{R}\right]\right] \Gamma_k;$$

$$k = 0, \ldots, N-1; \; \Gamma_0 = 0.$$

Proof: The conclusion is trivially true for $i = 0$. Assume it is true for epoch i. Then it is primarily a matter of computation to express $\tilde{x}_{i+1}$, v_{i+1}, and $\bar{\lambda}_{i+1}$ as explicit functions of $\tilde{x}_i$, v_i, $\bar{\lambda}_i$, u_i, and z_{i+1} via Eqs. (7), (8), (10), (11), (16), and (17), obtain the partial derivatives, apply the results in Appendix C for Lipschitz conditions of composite functions, and use the inequalities involved in the definition of G_{i+1} and the fact that the definitions of G_i and G_{i+1} imply that every initial segment of a sequence pair in G_{i+1} at epoch i is an element of G_i to show that inequalities (18) to (20) hold under the conditions of Lemma 1 for epoch i+1 with

$$\Gamma_{i+1} = 1 + \frac{4b^2}{r_{i+1}} + \left[\left[2 + \frac{2}{r_{i+1}}\right](|f_i|^3 + 3q_i) + 6\frac{b^2}{r_{i+1}} + \frac{4b^2}{r_{i+1}}\left[1 + |f_i| + 3\frac{b}{r_{i+1}}\right] + \frac{1}{r_{i+1}^2}\left[|f_i|^3 + 3q_i + \frac{6b^2}{r_{i+1}}\right]\right]\Gamma_i,$$

at least if h is less than each of some finite set of positive values which are needed to imply various computational inequalities but are independent of i. Using the inequalities for $|q_i|$, $|f_i|$, and $|r_{i+1}|$ in the problem formulation, we can then show that the conclusion of Lemma 1 holds at epoch i+1 for h^* equal to the minimum of this finite set and $\bar{h}$ of the preceding section, which is a positive constant independent of i. Hence, Lemma 1 follows by induction on i. □

4. APPROXIMATION OF CONDITIONAL STATE DENSITY

In order to develop an induction argument, we next consider a generic epoch $i \leqslant N - 1$, delete the subscripts of

$$\lambda_i, x_i, u_i, f_i, \psi_i, \tilde{x}_i, v_i, \eta_i, z_{i+1}, \Omega_i, c_i, D_i, M_i,$$

the last four to be defined shortly, suppress conditioning on Z_i in the rotation, and denote

$$\bar{p}_{\frac{x-\hat{x}}{\sqrt{v}}}(t) = \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\left[1 + \frac{1}{3}\eta(t^3 - 3t)\right]$$

and

$$\alpha(t) = \sqrt{2\pi}\, e^{\frac{t^2}{2}}\left[p_{\frac{x-\hat{x}}{\sqrt{v}}}(t) - \bar{p}_{\frac{x-\hat{x}}{\sqrt{v}}}(t)\right],$$

if the density $p_{\frac{x-\hat{x}}{\sqrt{v}}}$ exists at t.

4.1 Assumptions at Epoch i

For the purposes of Sections 4.2 and 4.3, the following four conditions are assumed for epoch i:

Assumption 1: For all t such that $|t| < M$, where $\frac{1}{4}h^{-1/8} \leqslant M \leqslant \frac{1}{4}h^{-1/4}$, $p_{\frac{x-\hat{x}}{\sqrt{v}}}(t)$ exists (at least as a Radon-Nikodym derivative of the probability distribution function of $\frac{x-\hat{x}}{\sqrt{v}}$, which itself is clearly well-defined since only Borel-measurable functions of Gaussian random variables ever appear here), and

$$|\alpha(t)| < \Omega h^2 e^{|t|}, \text{ where } \Omega < h^{-1/4}.$$

Assumption 2: $\Pr\left\{\left|\frac{x-\hat{x}}{\sqrt{v}}\right| > t\right\} < ke^{-\frac{1}{2}t^{2/k}}, \quad k = 2i+1, \quad \text{for all } t > \frac{1}{8}h^{-1/8}.$

Assumption 3: $(Z_i, Y_{i-1}) \in G_i$ and $(Z_i * z, Y_{i-1} * u) \in G_{i+1}$, where $*$ denotes concatenation.

Assumption 4: If $(Z_i, Y_{i-1})' \in G_i$ and differs from (Z_i, Y_{i-1}) in at most one component, whose values are denoted respectively by ρ' and ρ, then

$$|\alpha(t,\rho) - \alpha(t,\rho')| < e^{2|t|}(Ch^3 + Dh^2|\rho - \rho'|) \text{ if } |t| < M$$

and

$$|\alpha(t,\rho) - \alpha(s,\rho)| < e^{2\max(|t|,|s|)}(Ch^3 + Dh^2|s - t|) \text{ if } |s|, |t| < M,$$

where $C < h^{-1/4}$ and $D < h^{-1/4}$.

4.2 Linear Transformation

Applying the well-known formulas for the transformation of the probability density for a function of a random variable to $\frac{1}{\sqrt{m}}(y - \hat{y})$, where

$$y = fx + u,$$
$$\hat{y} = f\hat{x} + u,$$

and

$$m = f^2 v,$$

we can show that its density exists at t for $|t| < M$ and that there

$$p_{\frac{y-\hat{y}}{\sqrt{m}}}(t) = p_{\frac{x-\hat{x}}{\sqrt{v}}}(t).$$

Hence,

$$\alpha(t) = \left\{\sqrt{2\pi}\, e^{\frac{t^2}{2}} p_{\frac{y-\hat{y}}{\sqrt{m}}}(t) - \left[1 + \frac{1}{3}\eta(t^3 - t)\right]\right\} \triangleq \bar{\alpha}(t).$$

Lemma 2: In the preceding context, Assumptions 1, 2, and 4 of Section 4.1 hold with x, $\hat{x}$, v, α, and Y_{i-1} replaced respectively by y, $\hat{y}$, v, $\bar{\alpha}$, and $Y_{i-1} * u$, where $(Z_i, Y_{i-1} * u)'$ must be such that $(Z_i, Y_{i-1})' \in G_i$.

Proof: Assumptions 1 and 2 are established by the preceding remark and a few simple algebraic manipulations. Assumption 4 follows from the fact that $\bar{\alpha}$ is independent of u, from Assumption 4 for α, and from the fact that $\bar{\alpha} = \alpha$.

4.3 Addition of Process Noise

Now let

$$y = x + (1 + \psi x)\, w; \quad w = \frac{w_i}{f} \tag{21}$$

and define the random variable

$$s = \frac{y - \hat{x}}{\sqrt{m}},$$

where

$$m = \frac{m_{i+1}}{f_i^2} = v + q(1 + \psi\hat{x})^2\,; \quad q = \frac{q_i}{f^2}.$$

Then, since $|\psi\hat{x}| < 1$ by the assumptions of Section 4.1,

$$s = t + \sqrt{\frac{q}{m}}\,(1 + \psi\hat{x})\left|1 + \frac{\psi\sqrt{m}}{1 + \psi\hat{x}}t\right|\omega,$$

where ω is a normal (0,1) random variable and

$$p(t) = \frac{e^{-\frac{t^2}{2\epsilon^2}}}{\sqrt{2\pi\epsilon^2}}\left[1 + \frac{\eta}{3\epsilon^3}(t^3 - 3\epsilon^2 t) + \alpha\left(\frac{t}{\epsilon}\right)\right] \tag{22}$$

for $|t| \leqslant \epsilon M$, where $\epsilon = \sqrt{\frac{v}{m}}$. Let

$$\sigma = \sqrt{\frac{q}{m}\,(1 + \psi\hat{x})^2},$$

$$r = \left\{1 + \frac{\psi\sqrt{m}}{1 + \psi\bar{x}}\, t\right\}\omega, \tag{23}$$

$$g = \sigma\, r, \tag{24}$$

and

$$L = \epsilon M.$$

At this point, we also assume that $q_i \neq 0$, so $\sigma > 0$. If $q_i = 0$, Lemmas 3 to 5, to be proved later in this section, are trivially true.

Since $s = t + g$, it follows that for any $\Delta > 0$

$$\Pr\{s \in [\theta,\ \theta + \Delta)\} = \Pr\{(t + g) \in [\theta,\ \theta + \Delta) \text{ and } |t| \geqslant L\}$$
$$+ \Pr\{(t + g) \in [\theta, \theta + \Delta) \text{ and } |t| < L\}. \tag{25}$$

We partition the first event in Eq. (25) at

$$\bigcup_{k=0}^{\infty} (E_k \cup F_k),$$

where

$$E_k = \{t,g\colon (t + g) \in [\theta, \theta + \Delta) \text{ and } t \in (-L - (k + 1)\Delta,\ -L - k\Delta)\}$$

and

$$F_k = \{t,g\colon (t + g) \in [\theta, \theta + \Delta) \text{ and } t \in [L + k\Delta,\ L + (k + 1)\Delta)\}.$$

Now,

$$E_k \subset \{t,g\colon t \in A,\ g \in B\},$$

where

$$A = (-L - (k + 1)\Delta,\ -L - k\Delta]$$

and

$$B = [\theta + L + k\Delta,\ \theta + L + (k + 2)\Delta),$$

so

$$\Pr\{E_k\} \leqslant \Pr\{g \in B/t \in a\} \cdot \Pr\{t \in A\}.$$

For nonzero ξ, the density $p_{g/t}(\xi)$ exists such that

$$p_{g/t}(\xi) = \begin{cases} \dfrac{e^{-\frac{\xi^2}{2\sigma^2(1 + \phi t)^2}}}{\sqrt{2\pi\sigma^2(1 + \phi t)^2}} & \text{if } t \neq \dfrac{1}{\phi}, \quad \phi \triangleq \dfrac{\psi\sqrt{m}}{1 + \psi\bar{x}}; \\ 0 \text{ if } t = \dfrac{1}{\phi}. & \end{cases}$$

For a given ξ, this density is maximized over all t if

$$\sigma(1 + \phi t)^2 = \xi^2,$$

in which case

$$p_{g/t}(\xi) = \frac{1}{|\xi|\sqrt{2\pi e}}.$$

Hence, since all the events and probability measures are well-defined,

$$\begin{aligned}\Pr\{g \in B/t \in A\} &= \int_{t \in A} \Pr\{g \in B/t\}\, dP(t/t \in A) \\ &\leqslant \int_{t \in A} [2\Delta \max_{\xi \in B} p_{g/t}(x)]\, dP(t/t \in A) \\ &\leqslant \int_{t \in A} \frac{2\Delta}{\sqrt{2\pi e}}\, dP(t/t \in A) \quad \text{if } \theta \geqslant -L+1, \quad \text{by construction of } B \\ &\leqslant \frac{2\Delta}{\sqrt{2\pi e}} \int_{t \in A} dP(t/t \in A) = \frac{2\Delta}{\sqrt{2\pi e}}.\end{aligned}$$

Therefore,

$$\Pr\{E_k\} \leqslant \frac{2\Delta}{\sqrt{2\pi e}} \Pr\{t \in A\} \quad \text{if } \theta \geqslant -L+1.$$

Similarly,

$$\Pr\{F_k\} \leqslant \frac{2\Delta}{\sqrt{2\pi e}} \Pr\{t \in [L + k\Delta,\ L + (k+1)\Delta)\} \quad \text{if } \theta \leqslant L-1.$$

For $|\theta| \leqslant L - 1$, it follows from the definitions of E_k and F_k that

$$\begin{aligned}\Pr\{S \in [\theta, \theta+\Delta) \text{ and } |t| \geqslant L\} &= \Pr\{\bigcup_{k=0}^{\infty} (E_k \cup F_k)\} \\ &= \sum_{k=0}^{\infty} [\Pr\{E_k\} + \Pr\{F_k\}] \\ &\leqslant \frac{2\Delta}{\sqrt{2\pi e}} \sum_{k=0}^{\infty} [\Pr\{t \in (-L-(k+1)\Delta, -L-k\Delta]\} \\ &\quad + \Pr\{t \in [L+k\Delta, L+(k+1)\Delta)\}] \\ &\leqslant \frac{2\Delta}{\sqrt{2\pi e}} [\Pr\{t \in \bigcup_{k=0}^{\infty} (-L-(k+1)\Delta, -L-k\Delta]\} \\ &\quad + \Pr\{t \in \bigcup_{k=0}^{\infty} [L+k\Delta, L+(k+1)\Delta)\}] \\ &\qquad \text{(since the } k\text{-indexed intervals are all disjoint)} \\ &= \frac{2\Delta}{\sqrt{2\pi e}} \Pr\{|t| \geqslant L\} \quad \text{by construction} \\ &< \frac{\Delta}{2} \Pr\{|t| \geqslant L\}.\end{aligned}$$

It is shown in Appendix D that this inequality implies that the measure on the real line induced by the quasi-distribution function

$$F_s(\theta, \bar{L}) = \Pr\{s < \theta \text{ and } |t| \geqslant L\}$$

is absolutely continuous with respect to Lebesgue measure, so that $F_s(\theta, \bar{L})$ has at least a Radon-Nikodym derivative $p_s(\theta, \bar{L})$, and such a derivative exists with

$$0 \leqslant p_s(\theta, \bar{L}) < \Pr\{|t| \geqslant L\}$$

for $|\theta| \leqslant L - 1$. Using the usual abbreviations for probability density notation, we have

$$p(s, \bar{L}) < \Pr\{|t| \geqslant L\} \leqslant 2\Omega h^2 \sqrt{e}\, \Phi(M-1) + \Phi(M) + \int_M^{\infty} \xi^3 \frac{e^{-\frac{\xi^2}{2}}}{\sqrt{2\pi}}\, d\xi, \tag{26}$$

where Φ denotes the tail of the standard normal distribution, since

$$|t| \geqslant L \Leftrightarrow \left|\frac{x - \tilde{x}}{\sqrt{\nu}}\right| \geqslant M.$$

It is well known that for $M > 0$

$$\Phi(M) = \int_M^\infty \frac{e^{-\frac{\xi^2}{2}}}{\sqrt{2\pi}}\, d\xi < \frac{1}{M} \frac{e^{-\frac{M^2}{2}}}{\sqrt{2\pi}},$$

and it follows from repeated integration by parts that

$$\int_M^\infty \xi^3 e^{-\frac{\xi^2}{2}}\, d\xi = (M^2 + 2)\, e^{-\frac{M^2}{2}}.$$

Using these inequalities in Eq. (26), we can show that

$$p(s,\bar{L}) < \frac{e^{-\frac{M^2}{2}}}{\sqrt{2\pi}} \left[\frac{2\Omega h^2 e^M + 1}{M - 1} + M^2 + 2\right]$$

$$= \frac{e^{-\frac{s^2}{2} + |s|}}{\sqrt{2\pi}} \left[\left(\frac{2\Omega h^2}{M - 1}\right) e^{\frac{1}{2}(|s|^2 - M^2 + 2M - 2|s|)} + \left(M^2 + 2 + \frac{1}{M - 1}\right) e^{\frac{1}{2}(s^2 - M^2) - |s|}\right]$$

$$= \frac{e^{-\frac{s^2}{2} + |s|}}{\sqrt{2\pi}} \left\{\left(\frac{2\Omega h^2}{M - 1}\right) e^{-\frac{1}{2}[(M + |s| - 2)(M - |s|)]} + \left(M^2 + 2 + \frac{1}{M - 1}\right) e^{-\frac{1}{2}(M^2 - s^2) - |s|}\right\}.$$

With $M > \frac{1}{4} h^{-1/8}$, it follows for sufficiently small positive h and for $|s| < \epsilon M - 6\sqrt{\ln(1/h)}$ that

$$p(s,\bar{L}) < \frac{e^{-\frac{s^2}{2} + |s|}}{\sqrt{2\pi}} \left[\Omega h^2 e^{-8\ln(1/h)} + (M^2 + 3)\, e^{-8\ln(1/h)}\right]$$

$$< \frac{e^{-\frac{s^2}{2} + |s|}}{\sqrt{2\pi}} [\Omega h^{10} + h^7(1 + 3h)], \text{ since } M < h^{-1/4}. \tag{27}$$

Since t has a probability density function (in the sense of a Radon-Nikodym derivative of its distribution function) by assumption for arguments less in magnitude than L, the second event in Eq. (25) can be evaluated as

$$\Delta \int_{\frac{s-L}{\sigma}}^{\frac{s+L}{\sigma}} p_t\, (s - \sigma r)\, p_{r/t}\, (r,\, s - \sigma r)\, d\lambda_r,$$

where λ_r denotes Lebesgue measure for the r variable, since $s = t + \sigma r$ and the conditional density of r given t is well-defined. Hence the results of Appendix D also show that the quasi-distribution function

$$F_s\,(\theta, L) = \Pr\{s < \theta \text{ and } |t| < L\}$$

has a Radon-Nikodyn derivative $p_s\,(\theta,\, L)$ such that

$$p(s,L) = \int_{\frac{s-L}{\sigma}}^{\frac{s+L}{\sigma}} p_t\, (s - \sigma r)\, p_{r/t}\, (r, s - \sigma r)\, d\lambda_r \tag{28}$$

and therefore that s itself has a probability density in this sense (for arguments of magnitude less than $L-1$) such that

$$p_s(\theta) = p_s(\theta, \bar{L}) + p_s(\theta, L)$$

or, in abbreviated notation,

$$p(s) = p(s, \bar{L}) + p(s, L).$$

The conditional density for r in Eq. (28) is

$$p(r/t) = \frac{e^{-\frac{r^2}{2(1+\phi t)^2}}}{\sqrt{2\pi(1+\phi t)^2}}\ ;\ \phi = \frac{\psi\sqrt{m}}{1+\psi\bar{x}}. \tag{29}$$

To approximate this density in Eq. (28), denote

$$f(x) = \frac{e^{-\frac{r^2}{2(1+x)^2}}}{\sqrt{2\pi(1+x)^2}} \quad \text{(or zero if } x = -1\text{)},$$

where x now just denotes a real variable. Then, for $r \neq 0$,

$$f'(x) = \frac{f(x)}{1+x}\left[\left[\frac{r}{1+x}\right]^2 - 1\right] \quad \text{(or zero if } x = -1\text{)}$$

and

$$f''(x) = \frac{f(x)}{(1+x)^2}\left[\left[\frac{r}{1+x}\right]^4 - 5\left[\frac{r}{1+x}\right]^2 + 2\right] \quad \text{(or zero if } x = -1\text{)}.$$

For $r \neq 0$, these derivatives are continuous for all x, so Taylor's theorem with remainder implies that

$$f(x) = f(0) + xf'(x) + \frac{1}{2}x^2 f''(\theta),$$

for some θ in the union of intervals $[x, 0] \cup [0, x]$, where $[0, x]$, is regarded as empty if $x < 0$, etc. For $r = 0$, this argument still applies if $|x| < 1$.

Therefore, for $|s| < |\frac{1}{\phi}|$, using $\phi(s - \sigma r)$ to play the role of x we get

$$\begin{aligned} p_{r/t}(r, s-\sigma r) = {} & \frac{e^{-\frac{r^2}{2}}}{\sqrt{2\pi}}\left[1 + \phi(s-\sigma r)(r^2 - 1)\right] \\ & + \frac{1}{2}\phi^2\left[\frac{s-\sigma r}{1+\phi(s-\sigma\theta)}\right]^2 \left\{\frac{e^{-\frac{r^2}{2[1+\phi(s-\sigma\theta)]^2}}}{\sqrt{2\pi}[1+\phi(s-\sigma\theta)]^2}\right\} \\ & \times\left\{\left[\frac{r}{1+\phi(s-\sigma\theta)}\right]^4 - 5\left[\frac{r}{1+\phi(s-\sigma\theta)}\right]^2 + 2\right\}, \end{aligned} \tag{30}$$

where θ now denotes some value in $\left[r, \frac{s}{\sigma}\right] \cup \left[\frac{s}{\sigma}, r\right]$ which might vary with r and s. Applying Taylor's theorem for a first-order expansion of $f(x)$ likewise gives

$$p_{r/t}(r, s-\sigma r) = \frac{e^{-\frac{r^2}{2}}}{\sqrt{2\pi}} + \phi\left[\frac{s-\sigma r}{1+\phi(s-\sigma\theta)}\right]$$

$$\times \frac{e^{-\frac{r^2}{2[1+\phi(s-\sigma\theta)]^2}}}{\sqrt{2\pi}[1+\phi\,(s-\sigma\theta)]^2}\left\{\left[\frac{r}{1+\phi\,(s-\sigma\theta)}\right]^2-1\right\} \tag{31}$$

for $|s| < \left|\frac{1}{\phi}\right|$, where the value of θ is generally distinct from that of Eq. (30) but in the same range. Substituting Eqs. (22), (30), and (31) in Eq. (28), and changing the variable of integration to $u = s - \sigma r$, gives the following for $|s| < |\frac{1}{\phi}|$:

$$p(s,L) = \int_{-L}^{L} \frac{e^{-\frac{u^2}{2\epsilon^2}}}{\sqrt{2\pi\epsilon^2}} \left[\frac{e^{-\frac{(s-u)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\right] \left[1+\frac{\eta}{3\epsilon^3}\,(u^3-3\epsilon^2 u)\right]\left\{1+\phi u\left[\left[\frac{s-u}{\sigma}\right]^2-1\right]\right\} du$$

$$+ \int_{-L}^{L} \frac{e^{-\frac{u^2}{2\epsilon^2}}}{\sqrt{2\pi\epsilon^2}} \left[\frac{e^{-\frac{(s-u)^2}{2\sigma^2(1+\phi\beta)^2}}}{\sqrt{2\pi\sigma^2\,(1+\phi\beta)^2}}\right] \left[1+\frac{\eta}{3\epsilon^3}\,(u^3-3\epsilon^2 u)\right]\left[\frac{\phi^2 u^2}{2}\right]$$

$$\times \left\{\left[\frac{s-u}{\sigma(1+\phi\beta)}\right]^4 - 5\left[\frac{s-u}{\sigma(1+\phi\beta)}\right]^2+2\right\} du + \int_{-L}^{L} \alpha\left[\frac{u}{\epsilon}\right]\left[\frac{e^{-\frac{u^2}{2\epsilon^2}}}{\sqrt{2\pi\epsilon^2}}\right]$$

$$\times \left\{\frac{e^{-\frac{(s-u)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} + \phi u\left[\frac{e^{-\frac{(s-u)^2}{2\sigma^2(1+\phi\gamma)^2}}}{\sqrt{2\pi\sigma^2\,(1+\phi\gamma)^2}}\right]\left[\left[\frac{s-u}{\sigma\,(1+\phi\gamma)}\right]^2-1\right]\right\} d\lambda_u, \tag{32}$$

where β and $\gamma \in [u, 0] \cup [0, u]$ and vary with u and s. Completing squares in the exponents of Eq. (32) and using the fact that $\sigma^2 + \epsilon^2 = 1$ by definition gives, after some manipulation,

$$p(s,L) = \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\, I_1 + \frac{1}{2}\,\phi^2\, I_2 + E, \tag{33}$$

where

$$I_1 = \int_{-L}^{L} \frac{e^{-\frac{(u-\epsilon^2 s)^2}{2\sigma^2\epsilon^2}}}{\sqrt{2\pi\sigma^2\epsilon^2}} \left[1+\frac{\eta}{3\epsilon^3}\,(u^3-3\epsilon^2 u)\right] \left\{1+\phi u\left[\left[\frac{s-u}{\sigma}\right]^2-1\right]\right\} du, \tag{34}$$

$$I_2 = \int_{-L}^{L} \frac{e^{-\frac{u^2}{2\epsilon^2}}}{\sqrt{2\pi\epsilon^2}}\, \frac{e^{-\frac{(s-u)^2}{2\sigma^2(1+\phi\beta)^2}}}{\sqrt{2\pi\sigma^2\,(1+\phi\beta)^2}} \left[1+\frac{\eta}{3\epsilon^3}\,(u^3-3\epsilon^2 u)\right]$$

$$\times \left\{\left[\frac{s-u}{\sigma\,(1+\phi\beta)}\right]^4 - 5\left[\frac{s-u}{\sigma\,(1+\phi\beta)}\right]^2+2\right\} u^2 du, \tag{35}$$

and

$$E = \int_{-L}^{L} \frac{e^{-\frac{u^2}{2\epsilon^2}}}{\sqrt{2\pi\epsilon^2}} \left\{ \frac{e^{-\frac{(s-u)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} + \phi u \, \frac{e^{-\frac{(s-u)^2}{2\sigma^2(1+\phi\gamma)^2}}}{\sqrt{2\pi\sigma^2(1+\phi\gamma)^2}} \left[\left(\frac{s-u}{\sigma(1+\phi\gamma)} \right)^2 - 1 \right] \right\} \alpha \left(\frac{u}{\epsilon} \right) d\lambda_u. \quad (36)$$

Approximation of I_1

Let t now denote a new variable of integration in Eq. (34) such that

$$t = \frac{u - \epsilon^2 s}{\sigma\epsilon}.$$

Then $u = \epsilon\,(\sigma t + \epsilon s)$, $\dfrac{s-x}{\sigma} = \sigma s + \epsilon t$, and

$$I_1 = \int_{\frac{-L-\epsilon s}{\sigma}}^{\frac{L-\epsilon s}{\sigma}} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} [1 + c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + c_6 t^6] dt, \quad (37)$$

where

$$c_0 = \epsilon^2 \left(\frac{\eta\epsilon}{3} + \phi\sigma^2 \right) s^3 - \epsilon(\eta + \phi\epsilon)s + \frac{\phi\eta\epsilon^3}{3} [\epsilon^2\sigma^2 s^6 - (1 + 2\sigma^2)s^4 + 3s^2],$$

$$c_1 = \epsilon\sigma[\eta\epsilon + (3\sigma^2 - 2)\phi]s^2 - \sigma(\eta + \phi\epsilon) - \frac{\phi\eta\epsilon^2\sigma}{3} [(6\sigma^4 - 8\sigma^2 + 2)s^5 + (8\sigma^2 - 2)s^3 - 6s],$$

$$c_2 = [\eta\epsilon\sigma^2 + \phi\epsilon^2(1 - 3\sigma^2)]s + \frac{\phi\eta\epsilon}{3} [\epsilon^2(15\sigma^4 - 10\sigma^2 + 1)s^4 - 3(2\sigma^2 - 1)^2 s^2 + 3\sigma^2],$$

$$c_3 = \frac{\sigma}{3} (\eta\sigma^2 + 3\phi\epsilon^2) + \frac{2}{3}\phi\eta\sigma\epsilon^2[(10\sigma^4 - 10\sigma^2 + 2)s^3 + (4\sigma^2 - 3)s],$$

$$c_4 = \frac{1}{3} \phi\eta\epsilon\sigma^2[(15\sigma^4 - 20\sigma^2 + 3)s^2 + 2\sigma^2 - 3],$$

$$c_5 = \frac{2}{3} \phi\eta\sigma^3\epsilon^2(2 - 3\sigma^2)s, \text{ and}$$

$$c_6 = \frac{1}{3} \phi\eta\sigma^4\epsilon^2.$$

We now restrict s so that $|s| \leqslant L$ also, which implies $|\epsilon s| < L$, so applying standard results for moments of normal densities to Eq. (37) we obtain

$$I_1 = 1 + \frac{\epsilon^2}{3} (\eta\epsilon + 3\phi\sigma^2)(s^3 - 3s) + R_1 - R_2, \quad (38)$$

where

$$R_1 = \frac{\phi\eta\epsilon}{3} [\sigma^2\epsilon^2(s^6 - 12s^4 + 27s^2 - 6) + 3(1 - 3\sigma^2)s^2]$$

and

$$R_2 = \int_A \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} \left[1 + \sum_{n=0}^{6} c_n t^n\right] dt,$$

and where A denotes $\left[-\infty, \frac{L - \epsilon s}{\sigma}\right] \cup \left[\frac{L - \epsilon s}{\sigma}, \infty\right]$.

Since $0 \leqslant \sigma \leqslant 1$, $0 \leqslant \epsilon \leqslant 1$, and $\sigma^2 + \epsilon^2 = 1$ by their definitions, it is easy to show that $\sigma^2\epsilon^2 \leqslant 1/4$ and, after some manipulation, that

$$|R_1| < \left|\frac{\phi\eta\epsilon}{12}\right| (s + 2)^6$$

for *any* s. Applying inequality (E2) of Appendix E then gives

$$|R_1| < \left|\frac{\phi\eta\epsilon}{12}\right| \left(\frac{2^6}{e^4}\right) e^{|s|} \text{ for all } s. \tag{39}$$

Using the definition of ϕ and the bounds assumed for v, $\tilde{x}$, and m_{i+1}, we get

$$|R_1| < \frac{b}{F^2}\left(\frac{2}{e}\right)^4 h^2. \tag{40}$$

Denoting $\frac{-L - \epsilon s}{\sigma}$ by τ and $\frac{L - \epsilon s}{\sigma}$ by μ for convenience, we have for all $\zeta \geqslant 0$

$$|s| < L - \zeta \Rightarrow \mu > \frac{\sigma}{\epsilon} L + \frac{\epsilon}{\sigma}\zeta > 0$$

and

$$\tau < -\frac{\sigma}{\epsilon} L - \frac{\epsilon}{\sigma}\zeta < 0.$$

Hence, integrating by parts gives

$$\begin{aligned} R_2 = \frac{1}{\sqrt{2\pi}} \Bigg[&(1 + c_0 + c_2 + 3c_4 + 15c_6)\left(\int_{-\infty}^{\tau} e^{-\frac{t^2}{2}} dt + \int_{\mu}^{\infty} e^{-\frac{t^2}{2}} dt\right) \\ &+ (c_1 + 2c_3 + 8c_5)\left(e^{-\frac{\mu^2}{2}} - e^{-\frac{\tau^2}{2}}\right) + (c_2 + 3c_4 + 15c_6)\left(\mu e^{-\frac{\mu^2}{2}} - \tau e^{-\frac{\tau^2}{2}}\right) \\ &+ (c_3 + 4c_5)\left(\mu^2 e^{-\frac{\mu^2}{2}} - \tau^2 e^{-\frac{\tau^2}{2}}\right) + (c_4 + 5c_6)\left(\mu^3 e^{-\frac{\mu^2}{2}} - \tau^3 e^{-\frac{\tau^2}{2}}\right) \\ &+ c_5\left(\mu^4 e^{-\frac{\mu^2}{2}} - \tau^4 e^{-\frac{\tau^2}{2}}\right) + c_6\left(\mu^5 e^{-\frac{\mu^2}{2}} - \tau^5 e^{-\frac{\mu^2}{2}}\right)\Bigg]. \end{aligned}$$

Denoting $\frac{\sigma}{\epsilon} L + \frac{\epsilon}{\sigma}\zeta$ by B, we obtain the bound

$$|R_2| < \frac{e^{-\frac{B^2}{2}}}{\sqrt{2\pi}} \Big[2c_6 B^5 + c_5 B^4 + 2(c_4 + 5c_6)B^3 + (c_3 + 4c_5)B^2$$

$$+2\,(c_2+3c_4+15c_6)B+\frac{2}{B}(1+c_0+c_2+3c_4+15c_6)\Bigg],$$

since

$$\int_\mu^\infty e^{-\frac{u^2}{2}}\,du<\frac{1}{\mu}\,e^{-\frac{\mu^2}{2}}.$$

Since only values of B that are large compared to $\ln(1/h)$ will be of concern here, it follows from the sizes of c_0 through c_6 that this last bound for $|R_2|$ varies essentially as

$$\sqrt{\frac{2}{\pi}}\,\frac{e^{-\frac{B^2}{2}}}{B}$$

for sufficiently small positive h. Since

$$B>3\sqrt{\ln(1/h)}\Rightarrow\sqrt{\frac{2}{\pi}}\,\frac{e^{-\frac{B^2}{2}}}{B}<\frac{h^3}{\sqrt{2\pi}}\text{ for }h<\frac{1}{\epsilon},$$

then

$$|R_2|<\frac{h^3}{\sqrt{2\pi}}<\frac{h^3}{2}\,e^{|s|}\tag{41}$$

for $B>3\sqrt{\ln(1/h)}$ and $h<\frac{1}{e}$. This bound on B is implied by the condition

$$|s|<L-6\sqrt{\ln(1/h)},$$

because if $\frac{\epsilon}{\sigma}<\frac{1}{2}$, then $B>\frac{1}{2}L$ (of order $h^{-1/4}$); otherwise $B>\frac{1}{2}(L-|s|)$.

Applying the triangle inequality and Eqs. (38), (40), and (41), we get

$$\left|I_1-\left[1+\frac{\epsilon^2}{3}(\eta\epsilon+3\phi\sigma^2)(s^3-3s)\right]\right|<\frac{b}{F^2}\,h^2e^{|s|}\tag{42}$$

for sufficiently small positive h.

Bound on $|I_2|$

For $h<b^{-4}$, $M<\frac{1}{4\sqrt{hb}}$ and $|\bar{x}|<\frac{1}{4h}$ by assumption. Since $L<M$, it follows that

$$|\phi|<\frac{4\sqrt{b}}{3}\,h$$

and

$$|u|<L\Rightarrow|\phi u|<\frac{1}{3}\sqrt{h}$$

under these conditions. Hence for sufficiently small positive h,

$$\frac{e^{-\frac{(s-u)^2}{2\sigma^2(1+\phi\beta)^2}}}{\sqrt{2\pi\sigma^2(1+\phi\beta)^2}}<\frac{e^{-\frac{(s-u)^2}{2\sigma^2(1+\frac{1}{3}\sqrt{h})^2}}}{\sqrt{2\pi\sigma^2(1-\frac{1}{3}\sqrt{h})^2}}\tag{43}$$

if $\beta \in [u,0] \cup [0,u]$, and from Eq. (35),

$$|I_2| < \int_{-L}^{L} \frac{e^{-\frac{u^2}{2\epsilon^2}}}{\sqrt{2\pi\epsilon^2}} \left[\frac{e^{-\frac{(s-u)^2}{2\sigma^2(1+\frac{1}{3}\sqrt{h})^2}}}{\sqrt{2\pi\sigma^2(1-\frac{1}{3}\sqrt{h})^2}} \right] \left| 1 + \frac{\eta}{3\epsilon^3}(u^3 - 3u) \right|$$

$$\times \left[2\left(\frac{s-u}{\sigma}\right)^4 + 6\left(\frac{s-u}{\sigma}\right)^2 + 2 \right] u^2 \, du. \tag{44}$$

Completing the square in the exponent of Eq. (44) and using the fact that $\epsilon^2 + \sigma^2 = 1$, we get

$$|I_2| < \frac{1+\frac{1}{3}\sqrt{h}}{1-\frac{1}{3}\sqrt{h}} \left[\frac{e^{-\frac{s^2}{2(1+\frac{2}{3}\sigma^2\sqrt{h}+\frac{1}{9}\sigma^2 h)}}}{\sqrt{2\pi(1+\frac{2}{3}\sigma^2\sqrt{h}+\frac{1}{9}\sigma^2 h)}} \right]$$

$$\times \int_{-L}^{L} \left[\frac{e^{-\frac{(u-\bar{u})^2}{2\nu}}}{\sqrt{2\pi\nu}} \right] \left| 1 + \frac{\eta}{3\epsilon^3}(u^3-3u) \right| \left[2\left(\frac{s-u}{\sigma}\right)^4 + 6\left(\frac{s-u}{\sigma}\right)^2 + 2 \right] u^2 \, du, \tag{45}$$

where

$$\bar{u} = \frac{\epsilon^2 s}{1+\frac{1}{3}\sigma^2\sqrt{h}\left(2+\frac{1}{3}\sqrt{h}\right)} \quad \text{and} \quad \nu = \frac{\epsilon^2\sigma^2\left(1+\frac{1}{3}\sqrt{h}\right)}{1+\frac{1}{3}\sigma^2\sqrt{h}\left(2+\frac{1}{3}\sqrt{h}\right)}$$

Now, for $s \geqslant 0$ and $k > 0$,

$$s < \left[\frac{6}{\sigma^2\sqrt{h}\,(2+\frac{1}{3}\sqrt{h})} + 2 \right] k \Rightarrow s < \frac{2k\left[3+\sigma^2\sqrt{h}\left(2+\frac{1}{3}\sqrt{h}\right)\right]}{\sigma^2\sqrt{h}\left(2+\frac{1}{3}\sqrt{h}\right)}$$

$$\Rightarrow s\left[1 - \frac{1}{1+\frac{1}{3}\sigma^2\sqrt{h}\left(2+\frac{1}{3}\sqrt{h}\right)}\right] < 2k$$

$$\Rightarrow -\frac{s^2}{2\left[1+\frac{1}{3}\sigma^2\sqrt{h}\left(2+\frac{1}{3}\sqrt{h}\right)\right]} < -\frac{s^2}{2} + ks.$$

W

Since $\sigma^2 \leqslant 1$,

$$|s| < \frac{2k}{\sqrt{h}} \Rightarrow \frac{s^2}{2\left[1+\frac{1}{3}\sigma^2\sqrt{h}\left(2+\frac{1}{3}\sqrt{h}\right)\right]} < -\frac{s^2}{2} + k|s|$$

for both positive and negative s for all sufficiently small positive h. Using this result for $k = 1/8$ in Eq. (45) leads to the bound

$$|I_2| < \frac{2\epsilon^2}{1-\frac{1}{3}\sqrt{h}}\left(\frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\right) e^{\frac{|s|}{8}} \int_{-\infty}^{\infty} \frac{e^{-\frac{(u-\bar{u})^2}{4\epsilon^2\sigma^2}}}{\sqrt{4\pi\epsilon^2\sigma^2}}$$

$$\times \left|1+\frac{\eta}{3}\left(\frac{u^3}{\epsilon^3}-3\frac{u}{\epsilon}\right)\right|\left[2\left(\frac{s-u}{\sigma}\right)^4+6\left(\frac{s-u}{\sigma}\right)^2+2\right]\frac{u^2}{\epsilon^2}\,du \tag{46}$$

for sufficiently small positive h, since

$$|s| < L \Rightarrow |s| < \frac{1}{4\sqrt{h}} \Rightarrow |s| < \frac{2(1/8)}{\sqrt{h}}.$$

Since $|\eta| < 3h\sqrt{b}$ by assumption, the integral in Eq. (46) is bounded by

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{(u-\bar{u})^2}{4\epsilon^2\sigma^2}}}{\sqrt{4\pi\sigma^2\epsilon^2}}\left[1+h\sqrt{b}\left(\left|\frac{u}{\epsilon}\right|^3+3\left|\frac{u}{\epsilon}\right|\right)\right]\left[2\left|\frac{s-u}{\sigma}\right|^4+6\left|\frac{s-u}{\sigma}\right|^2+2\right]\left|\frac{u}{\epsilon}\right|^2 du$$

by the triangle inequality and elementary properties of integrals. Changing the variable of integration in (46) to

$$t = \frac{u-\bar{u}}{\sigma\epsilon\sqrt{2}}$$

leads to the inequality

$$|I_2| < \left(\frac{e^{-\frac{s^2}{2}+\frac{|s|}{8}}}{\sqrt{2\pi}}\right) 8\sqrt{2}\int_0^{\infty} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\left\{1+h\sqrt{2b}\left[\left(t+|s|\right)^3+3(t+|s|)\right]\right\}$$

$$\times\left[8(t+|s|)^4+12(t+|s|)^2+2\right](t+|s|)^2\,dt \tag{47}$$

for sufficiently small positive h, since σ and $\epsilon \leqslant 1$, and since

$$\frac{u}{\epsilon} = \sqrt{2}\,\sigma t + \frac{\epsilon s}{1+\frac{1}{3}\sigma^2\sqrt{h}\left(2+\frac{1}{3}\sqrt{h}\right)}$$

and

$$\frac{s-u}{\sigma} = \sqrt{2}\epsilon t + \frac{\sigma\left(+\frac{1}{3}\sqrt{h}\right)^2 s}{1+\frac{1}{3}\sigma^2\sqrt{h}\left(2+\frac{1}{3}\sqrt{h}\right)},$$

so $$\left|\frac{u}{\epsilon}\right| \leqslant \sqrt{2}\,(|t|+|s|) \quad \text{and} \quad \left|\frac{s-u}{\sigma}\right| \leqslant \sqrt{2}(|t|+|s|)$$

for sufficiently small positive h. Also, for sufficiently small positive h, the bound of (47) is smaller than

$$\left(e^{-\frac{s^2}{2}+\frac{|s|}{8}}\right) 8\sqrt{2} \int_0^\infty \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} [512(t^6+s^6) + 192(t^4+s^4) + 8(t^2+s^2)]\, dt$$

by inequality (E2) of Appendix E, and by inequality (E1) of that appendix and standard results for normal moments, the integral in this expression is less than

$$256s^6 + 96s^4 + 4s^2 + 4132 < 256(|s|+2)^6 < \frac{(256)(7^6)}{e^4}\, e^{\frac{7}{8}|s|}.$$

Substituting back into inequality (47) and using the definition of $|\phi|$, we get

$$\left|\frac{1}{2}\,\phi^2\, I_2\right| < \left(\frac{8}{9}\right)(8\sqrt{2})\, \frac{(256)(117649)}{e^4}\, b\, h^2 \left[\frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\right] e^{|s|} \tag{48}$$

for sufficiently small positive h.

Bound on E

From Eq. (36)

$$\begin{aligned} E = \int_{-L}^{L} \alpha\left(\frac{u}{\epsilon}\right) \frac{e^{-\frac{u^2}{2\epsilon^2}}}{\sqrt{2\pi\epsilon^2}} \left[\frac{e^{-\frac{(s-u)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\right] d\lambda_u \\ + \phi \int_{-L}^{L} \alpha\left(\frac{u}{\epsilon}\right) \frac{e^{-\frac{u^2}{2\epsilon^2}}}{\sqrt{2\pi\epsilon^2}} \left[\frac{e^{-\frac{(s-u)^2}{2\sigma^2(1+\phi\gamma)^2}}}{\sqrt{2\pi\sigma^2(1+\phi\gamma)^2}}\right] \left\{\left[\frac{s-u}{\sigma(1+\phi\gamma)}\right]^2 - 1\right\} u\, d\lambda_u. \end{aligned} \tag{49}$$

By assumption $\left|\alpha\left(\frac{u}{\epsilon}\right)\right| < \Omega\, h^2 e^{|u/\epsilon|}$ for $|u| \leqslant L$. By the triangle inequality for integrals, the first of the two terms in Eq. (49) is bounded in magnitude by

$$\Omega\, h^2 \int_{-L}^{L} e^{|u/\epsilon|}\, \frac{e^{-\frac{u^2}{2\epsilon^2}}}{\sqrt{2\pi\epsilon^2}} \left[\frac{e^{-\frac{(s-u)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\right] du.$$

Completing the square in the exponent gives this bound as

$$\Omega\, h^2 \frac{e^{-\frac{s^2}{2}-\frac{\sigma^2}{2}}}{\sqrt{2\pi}} \left[e^{-\frac{\epsilon s}{2}} \int_{-L}^{0} \frac{e^{-\frac{(u-\epsilon^2 s-\sigma^2\epsilon)^2}{2\sigma^2\epsilon^2}}}{\sqrt{2\pi\sigma^2\epsilon^2}}\, du + e^{\frac{\epsilon s}{2}} \int_0^L \frac{e^{-\frac{(u-\epsilon^2 s+\sigma^2\epsilon)^2}{2\sigma^2\epsilon^2}}}{\sqrt{2\pi\sigma^2\epsilon^2}}\, du\right].$$

Since the integral of a normal density over a finite interval is less than unity, this bound clearly implies

$$\left|\int_{-L}^{L} \alpha\left(\frac{u}{\epsilon}\right) \frac{e^{-\frac{u^2}{2\epsilon^2}}}{\sqrt{2\pi\epsilon^2}} \left[\frac{e^{-\frac{(s-u)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\right] d\lambda_u\right| \leqslant 2\Omega\, h^2 \frac{e^{-s^2}}{\sqrt{2\pi}}\, e^{|s|}. \tag{50}$$

The reasoning used to establish inequality (41) likewise shows that

$$\frac{e^{-\frac{(s-u)^2}{2\sigma^2(1+\phi\gamma)^2}}}{\sqrt{2\pi\sigma^2(1+\phi\gamma)^2}} < \frac{e^{-\frac{(s-u)^2}{2\sigma^2(1+1/3\sqrt{h})^2}}}{\sqrt{2\pi\sigma^2\left[1+\frac{1}{3}\sqrt{h}\right]^2}} \tag{51}$$

for sufficiently small positive h. Hence, from the triangle inequality for integrals, the second term in Eq. (49) is bounded in magnitude by

$$2\Omega h^3 \frac{1+\sqrt{h}}{1-\sqrt{h}} \int_{-L}^{L} e^{|u/\epsilon|} \frac{e^{-\frac{u^2}{2\epsilon^2}}}{\sqrt{2\pi\epsilon^2}} \left[\frac{e^{-\frac{(s-u)^2}{2\sigma^2(1+\frac{1}{3}\sqrt{h})^2}}}{\sqrt{2\pi\sigma^2(1+\frac{1}{3}\sqrt{h})^2}}\right]\left[\left(\frac{s-u}{\sigma}\right)^2 - 1\right] udu. \tag{52}$$

Completing the square in the exponent gives the exponential factor in the integrand of Eq. (52) as

$$\sqrt{e}\left(\frac{e^{-\frac{[s-\epsilon\,\mathrm{sgn}(u)]^2}{2[1+\frac{1}{3}\sigma^2\sqrt{h}\,(2+\frac{1}{3}\sqrt{h})]}}}{\sqrt{2\pi[1+\frac{1}{3}\sigma^2\sqrt{h}\,(2+\frac{1}{3}\sqrt{h})]}}\right)\left[\frac{e^{-\frac{(u-\hat{u})^2}{2\nu}}}{\sqrt{2\pi\nu}}\right], \tag{53}$$

where

$$\hat{u} = -\epsilon\ \mathrm{sgn}\,(u) + \frac{\epsilon^2[s+\epsilon\ \mathrm{sgn}\,(u)]}{1+\frac{1}{3}\sigma^2\sqrt{h}\,(2+\frac{1}{3}\sqrt{h})}$$

and

$$\nu = \frac{\epsilon^2\sigma^2(1+\frac{1}{3}\sqrt{h})^2}{1+\frac{1}{3}\sigma^2\sqrt{h}\,(2+\frac{1}{3}\sqrt{h})}.$$

The first factor in brackets in Eq. (53) is bounded by

$$\frac{1+\sqrt{h}}{\sqrt{2\pi}}\, e^{\frac{(|s|+1)^2}{2(1+\sqrt{h})}}$$

for sufficiently small positive h. This bound is equal to

$$\frac{1+\sqrt{h}}{\sqrt{2\pi e}}\, e^{-\frac{\frac{1}{2}s^2+|s|}{1+\sqrt{h}}} \quad \text{or} \quad \frac{1+\sqrt{h}}{\sqrt{2\pi e}}\left(e^{-s^2+|s|}\right)e^{\frac{s^2\sqrt{h}}{2}}.$$

Since $s^2 < \frac{1}{16\sqrt{h}}$ by assumption, the first two factors in Eq. (53) are bounded uniformly in u by

$$e^{\frac{1}{16}}\,(1+\sqrt{h})\left(\frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\right)e^{|s|},$$

for sufficiently small positive h. The last factor in Eq. (53) is bounded by the sum

$$\frac{e^{-\frac{(u-\hat{u}_1)^2}{2\nu}}}{\sqrt{2\pi\nu}} + \frac{e^{-\frac{(u-\hat{u}_2)^2}{2\nu}}}{\sqrt{2\pi\nu}},$$

where

$$\hat{u}_1 = -\epsilon + \frac{\epsilon^2 (s+\epsilon)}{1 + \frac{1}{3}\sigma^2\sqrt{h}\,(2 + \frac{1}{3}\sqrt{h})}$$

and

$$\hat{u}_2 = \epsilon + \frac{\epsilon^2 (s-\epsilon)}{1 + \frac{1}{3}\sigma^2\sqrt{h}\,(2 + \frac{1}{3}\sqrt{h})}.$$

Hence, the second term in Eq. (49) is bounded in magnitude by

$$2\Omega h^3 \frac{(1+\sqrt{h})^3}{1-\sqrt{h}}\, e^{\frac{1}{16}} \left(\frac{e^{-\frac{s^2}{2}+|s|}}{\sqrt{2\pi}}\right) \left\{\int_{-\infty}^{\infty} \frac{e^{-\frac{(u-\hat{u}_1)^2}{2\nu}}}{\sqrt{2\pi\nu}} \left|\left(\frac{s-u}{\sigma}\right)^2 - 1\right| u\,du \right.$$
$$\left. + \int_{-\infty}^{\infty} \frac{e^{-\frac{(u-\hat{u}_2)^2}{2\nu}}}{\sqrt{2\pi\nu}} \left|\left(\frac{s-u}{\sigma}\right)^2 - 1\right| u\,du\right\}. \tag{54}$$

For either of the two integrals in Eq. (54), let

$$t = \frac{u-\hat{u}}{\sqrt{\nu}}, \quad \text{so } \frac{s-u}{\sigma} = \frac{s-\hat{u}}{\sigma} + \frac{\sqrt{\nu}}{\sigma}\,t.$$

Since

$$\hat{u} = \mp\epsilon\sigma^2\left[1 + \frac{\frac{1}{3}\sqrt{h}\,(2+\frac{1}{3}\sqrt{h})}{1 + \frac{1}{3}\sigma^2\sqrt{h}\,(2+\frac{1}{3}\sqrt{h})}\right] + \frac{e^2 s}{1 + \frac{1}{3}\sigma^2\sqrt{h}\,(2+\frac{1}{3}\sqrt{h})},$$

$$\frac{s-u}{\sigma} = \frac{\epsilon^2\sigma\,(1+\frac{1}{3}\sqrt{h})^2}{1 + \frac{1}{3}\sigma^2\sqrt{h}\,(2+\frac{1}{3}\sqrt{h})}\,t \pm \epsilon\sigma\left[\frac{1 + \frac{\sigma^2+1}{3}\sqrt{h}\,(2+\frac{1}{3}\sqrt{h})}{1 + \frac{1}{3}\sigma^2\sqrt{h}\,(2+\frac{1}{3}\sqrt{h})}\right]$$
$$+ \frac{s}{\sigma}\left\{\frac{\sigma^2[1 + \frac{1}{3}\sqrt{h}\,(2+\frac{1}{3}\sqrt{h})]}{1 + \frac{1}{3}\sigma^2\sqrt{h}\,(2+\frac{1}{3}\sqrt{h})}\right\},$$

so that, for sufficiently small positive h,

$$|u| = |\sqrt{\nu}\,t + \hat{u}| < 2(|s| + |t|) + 1$$

and

$$\left|\frac{s-u}{\sigma}\right| < 2\,(|s| + |t|) + 1.$$

Thus, each of the two integrals in Eq. (54) is bounded in magnitude for such h by

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\,[8(|s| + |t|)^3 + 12(|s| + |t|)^2 + (|s| + |t|)]\,dt$$

$$\leqslant \int_{-\infty}^{\infty} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\,[64\,(|s|^3 + |t|^3) + 48(s^2 + t^2) + |s| + |t|]\,dt,$$

by inequality (E2) of Appendix E. From standard results for moments of normal distributions, this last bound is

$$|4s|^3 + 3|4s|^2 + \frac{1}{4}|4s| + 48 + 129\sqrt{\frac{2}{\pi}}.$$

Since $|4s| \leqslant 4M \leqslant h^{-1/4}$, using this bound in Eq. (54) we can show that the second term in Eq. (49) is bounded in magnitude by

$$6\Omega\, h^{1/4}\, h^2\, \frac{e^{-\frac{s^2}{2}+|s|}}{\sqrt{2\pi}}$$

for sufficiently small positive h. With inequality (50), this also means that

$$|E| < (2 + 6h^{1/4})\Omega\, h^2\, \frac{e^{-\frac{s^2}{2}+|s|}}{\sqrt{2\pi}} < \frac{5}{2}\,\Omega\, h^2 \left(\frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\right) e^{|s|} \tag{55}$$

for all h less than some positive value.

Lemma 3: In the context of Eqs. (21) to (23), there exists an $h^* > 0$ which depends only on the parameters a and b of the unperturbed problem, such that if $h \leqslant h^*$ and if $|\theta| \leqslant \sqrt{\frac{f_i^2 v}{m_{i+1}}}\,M - 6\sqrt{\ln(1/h)}$, then $p_s(\theta)$ exists and

$$|\bar{\alpha}(\theta)| < [3\Omega + b(1/F^2 + 6,000,000)]\,e^{|\theta|},$$

where

$$\bar{\alpha}(\theta) = \sqrt{2\pi}\, e^{\frac{\theta^2}{2}}\,[p_s(\theta) - \bar{p}_s(\theta)]$$

and

$$\bar{p}_s(\theta) = \frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}}\,[1 + \frac{1}{3}\dot{\eta}_{i+1}\,(\theta^3 - 3\theta)].$$

Proof: From the definitions of ϵ, σ, and ϕ,

$$\dot{\eta}_{i+1} = \epsilon^3\eta_i + 3\phi\epsilon^2\sigma^2.$$

Since $|\bar{x}| \leqslant \ln(1/h)$ and $m = \dfrac{m_{i+1}}{f_i^2} > a$ by assumption,

$$|\theta| \leqslant \sqrt{\frac{f_i^2 \nu}{m_{i+1}}}\, M - 6\sqrt{\ln(1/h)} \Rightarrow |\theta| < M \Rightarrow |\theta| < \frac{1}{4} h^{-1/4}$$

$$\Rightarrow |\theta| < \frac{1}{2h\sqrt{a}} \text{ for sufficiently small } h > 0$$

$$\Rightarrow |\theta| < \left|\frac{1+\psi\bar{x}}{\psi\sqrt{m}}\right| - \frac{1}{|\phi|}.$$

The conclusion of the lemma then follows from the results of Eqs. (27), (33), (42), (48), and (55) and the fact that

$$p_s(\theta) = p_s(\theta, L) + p_s(\theta, \bar{L}),$$

because the condition of sufficiently small positive h was invoked in the derivation of these results only a finite number of times and in such a way each time that "sufficiently small" depended only on the values of a and b in the corresponding unperturbed problem. Thus h^* here can be chosen as either the smallest of these values which is still strictly positive or, if it is smaller, the maximum h for which

$$\frac{1}{4} h^{-1/4} \geqslant \frac{1}{2h\sqrt{a}},$$

which is also positive. □

Lemma 4: In the context of Eqs. (21) to (23), there exists an $h^* > 0$ which depends only on parameters a and b, such that for every $h \leqslant h^*$,

$$\Pr(s \geqslant \theta) \leqslant (k+1) e^{-\frac{1}{2}\theta^{\frac{2}{k+1}}} \text{ if } \theta \geqslant \frac{1}{4} h^{-1/8} \text{ and } k < h^{-1/12}, \text{ where } k = 2i.$$

Proof: Let

$$\bar{\theta} = \sqrt{\frac{f_i^2 \nu}{m_{i+1}}}\, M - 6\sqrt{\ln(1/h)}$$

and let $\tilde{h} > 0$ be such that $h \leqslant \tilde{h} \Rightarrow \bar{\theta} < \dfrac{1}{|\phi|}$ and $\bar{\theta} > \dfrac{1}{8} h^{-1/8}$(see the proof of Lemma 3). If $\dfrac{1}{4} h^{-1/8} \leqslant \theta < \bar{\theta}$, the conclusion of the lemma follows by subtraction from the integral of $\bar{p}_s$ and $\bar{\alpha}$ of Lemma 3 for sufficiently small positive h (the maximum allowable value depending only on parameters a and b), since $\dfrac{1}{k+1} \leqslant 1$. For $\theta \geqslant \bar{\theta} > \dfrac{1}{8} h^{-1/8}$, the event $\{\theta: |s| \geqslant \theta\}$ is contained in the union of the two events

$$E_1 = \{t: |t| > \theta^{\frac{k}{k+1}}\}$$

and

$$E_2 = \{t, g: |g| \geqslant \theta - \theta^{\frac{k}{k+1}} \text{ and } |t| \leqslant \theta^{\frac{k}{k+1}}\},$$

since $s = t + g$ by construction. Since $m \geqslant v \geqslant 0$,

$$|t| \geqslant \theta^{\frac{k}{k+1}} \Leftrightarrow \left|\frac{x-\tilde{x}}{\sqrt{v}}\right| \geqslant \sqrt{\frac{m}{v}}\,\theta^{\frac{k}{k+1}} \Rightarrow \left|\frac{x-\tilde{x}}{\sqrt{v}}\right| < \theta^{\frac{k}{k+1}}.$$

Hence,

$$\Pr\{E_1\} < ke^{-\frac{1}{2}\theta^{\frac{2}{k+1}}}$$

by Assumption 2 of Section 4.1. Also,

$$\Pr\{E_2\} = \int_{|t| \leqslant \theta^{\frac{k}{k+1}}} \Pr\{|g| \geqslant \theta - \theta^{\frac{k}{k+1}}/t\}\, dP(t),$$

but

$$\Pr\{|g| \geqslant \theta - \theta^{\frac{k}{k+2}}/t\} = 2\Phi\left(\frac{\theta - \theta^{\frac{k}{k+2}}}{\sigma|1+\phi t|}\right) \quad (\Phi = \text{normal tail as before})$$

$$\leqslant 2\Phi\left(\frac{\theta^{\frac{1}{k+1}} - 1}{4h^{\frac{1}{12}} + h}\right) \quad \left(\text{if } |t| \leqslant \theta^{\frac{k}{k+1}}, \text{ since } h \text{ is small enough that } \bar{\theta} > \frac{1}{8} h^{-1/8}\right)$$

$$\leqslant 2\Phi\left(\theta^{\frac{1}{k+1}}\right) \quad (\text{if } \theta > (1 - 4h^{1/12} - h)^{-k-1}).$$

This last condition is satisfied for $k < h^{-1/12}$ for sufficiently small positive h because

$$\theta > \bar{\theta} > \frac{1}{8} h^{-1/8} \text{ (for sufficiently small positive } h)$$

$$> e^6 > e^{5(k+1)h^{\frac{1}{12}}} > e^{-(k+1)\ln(1-5h^{\frac{1}{12}})}$$

$$= e^{\ln[(1-5h^{\frac{1}{12}})^{-k+1)}]} > (1 - 5h^{\frac{1}{12}})^{-(k+1)} > (1 - 4h^{\frac{1}{12}} - h)^{-(k+1)}$$

for sufficiently small positive h. Hence,

$$\Pr\{|g| \geqslant \theta - \theta^{\frac{k}{k+1}}/t\} < \frac{2}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}\theta^{\frac{2}{k+1}}}}{\theta^{\frac{2}{k+1}}} < \frac{1}{2} e^{-\frac{1}{2}\theta^{\frac{2}{k+1}}}$$

for h small enough that $\bar{\theta} \geqslant 1$. For $\theta \geqslant \bar{\theta}$, therefore,

$$\Pr\{E_2\} < \frac{1}{2} e^{-\frac{1}{2}\theta^{\frac{2}{k+1}}} \int_{|t| \leqslant \theta^{\frac{k}{k+1}}} dP(t) < e^{-\frac{1}{2}\theta^{\frac{2}{k+1}}}$$

under these conditions, and

$$\Pr\{|s| \geqslant \theta\} \leqslant \Pr\{E_1\} + \Pr\{E_2\} < (k+1)\, e^{-\frac{1}{2}\theta^{\frac{2}{k+1}}}$$

The lemma follows because $h^* > 0$ can be chosen on the minimum of the positive values $\bar{h}$ and the three positive maximum values of h needed for the inequalities used in this proof.□

Lipschitz Condition for R_1

From the definitions of ϕ, ϵ, and σ in terms of ν and $\tilde{x}$, from the Lipschitz conditions of Lemma 1 for ν, $\tilde{x}$, and $\bar{\lambda}$, and from the results of Appendix C it follows from partial differentiation of R_1, rewritten using Eqs. (14) and (30), as

$$R_1 = \frac{\phi\bar{\lambda}\epsilon}{3\nu^{3/2}} [\sigma^3\epsilon^2(s^6 - 12s^4 + 27s^2 - 6) + 3(1 - 3\sigma^2)s^2],$$

with respect to ϵ, σ, ϕ, λ, and s (using the magnitude bounds for η and ν assumed in Section 4.1) that

$$|R_1(s, \rho) - R_1(s', \rho)| \leqslant \frac{1}{4} e^{2\max(|s|, |s'|)} \sqrt{\ln(1/h)}\, h^2\, |s - s'| \tag{56}$$

and

$$|R_1(s, \rho) - R_1(s, \rho')| \leqslant \frac{1}{4} e^{2|s|} \sqrt{\ln(1/h)}\, h^2 |\rho - \rho'| \tag{57}$$

for sufficiently small positive h, where the arguments of R_1 are given in the context of Assumption 4 of Lemma 2, because $e^{|x|}$ dominates any polynomial in x for large enough x.

Lipschitz Condition for I_2

From Eq. (35),

$$I_2 = \int_{-L}^{L} \frac{e^{-\frac{u^2}{2\epsilon^2}}}{\sqrt{2\pi\epsilon^2}} \left\{\left[\frac{s - u}{\sigma(1 + \phi\beta)}\right]^4 - 5\left[\frac{s - u}{\sigma(1 + \phi\beta)}\right]^2 + 2\right\} u^2\, du$$

$$+ \frac{\eta}{3\epsilon^3} \int_{-L}^{L} \frac{e^{-\frac{u^2}{2\epsilon^2}}}{\sqrt{2\pi\epsilon^2}} (u^5 - 3\epsilon^2u^3) \left\{\left[\frac{s - u}{\sigma(1 + \phi\beta)}\right]^4 - 5\left[\frac{s - u}{\sigma(1 + \phi\beta)}\right]^2 + 2\right\} du, \tag{58}$$

where $\beta(u, s) \in [0, u] \cup [u, 0]$. It is clear from the computations for Eq. (47) that the second integral in Eq. (58) is bounded in magnitude by

$$\sqrt{\ln(1/h)}\; h\; e^{-\frac{s^2}{2} + |s|}$$

for sufficiently small positive h if $|s| < L$. Completing the square in the exponent gives the first integral in Eq. (58) as

$$\frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}} \int_{-L}^{L} \frac{e^{\left[\frac{\sigma^2\theta(2+\theta)s^2}{1+\sigma^2\theta(2+\theta)}\right]}}{\sqrt{1 + \sigma^2\theta(2 + \theta)}} \left[\frac{e^{-\frac{(u-\bar{u})^2}{2\nu}}}{\sqrt{2\pi\nu}}\right] u^2\left\{\left[\frac{s - u}{\sigma(1 + \theta)}\right]^4 - 5\left[\frac{s-u}{\sigma(1 + \theta)}\right]^2 + 2\right\} du, \tag{59}$$

where

$$\bar{u} = \frac{\epsilon^2 s}{1 + \sigma^2\theta(2 + \theta)},$$

$$\nu = \frac{\epsilon^2\sigma^2(1 + \theta)^2}{1 + \sigma^2\theta\,(2 + \theta)}, \text{ and}$$

$\theta(s, u, \rho) = \phi(\rho)\,\beta(s, u)$ (so $|\theta| < \sqrt{h}$ in the context of interest here).

If the variable of integration in Eq. (59) is changed to

$$x = \frac{u - \bar{u}}{\sqrt{\nu}},$$

then

$$u = \frac{\epsilon^2 s}{1 + \sigma^2 \phi\mu (2 + \phi\mu)} + \frac{\epsilon\sigma (1 + \phi\mu) x}{\sqrt{1 + \sigma^2 \phi\mu (2 + \phi\mu)}} \tag{60}$$

and

$$\frac{s - u}{\sigma (1 + \phi\mu)} = \frac{\sigma s}{(1 + \phi\mu)\,[1 + \sigma^2 \mu (2 + \phi\mu)]} - \frac{\epsilon x}{\sqrt{1 + \sigma^2 \phi\mu (2 + \phi\mu)}}, \tag{61}$$

with

$$\mu(s,\ x) = \beta[s,\ u(x)] \quad (\text{so } \theta = \phi\mu). \tag{62}$$

The partial derivative of this integral (denoted now as T) with respect to ϕ can be expressed by Leibnitz's rule. The resulting integral can then be bounded (by standard results for normal moments) as

$$1000 \times (\text{polynomial in } s,\ \phi),$$

since $s^2 < \frac{1}{16} h^{-1/2}$, which implies that $|e^{\theta s}| < 2$. Then by the definitions of the variables in terms of $\tilde{x}$ and v, by the Lipschitz conditions established in Lemma 1 for $\tilde{x}$ and v, and by the results of Appendix C, it follows by routine manipulation that

$$|T(\rho) - T(\rho')| < \frac{1}{4}\sqrt{\ln(1/h)}\ e^{|s|}\ |\rho - \rho'|,$$

with ρ and ρ' used in the content of Section 4.1, for sufficiently small positive h, since $e^{|s|}$ dominates any polynomial in s for large enough s.

For the case of fixed ρ and varying s, denote the integrand of T by H. The triangle inequality for integrals gives (for $s_1 \leqslant s_2$, and $|s_1|, |s_2| < L - 6\sqrt{\ln(1/h)}$)

$$|T(s_1) - T(s_2)| \leqslant \int_{L_1}^{L_2} \left\{ \max_{\mu(s), s \in [s_1, s_2]} \left\{ \left\| \frac{\partial H}{\partial \mu} \right\| \right\} |\mu(s_1) - \mu(s_2)| + \max_{s \in [s_1, s_2]} \left\{ \left\| \frac{\partial H}{\partial s} \right\| \right\} |s_1 - s_2| \right\} dx$$
$$+ \left| \int_{L_3}^{L_1} H\, dx \right| + \left| \int_{L_2}^{L_4} H\, dx \right|, \tag{63}$$

where $[L_1, L_2]$ is the common x range of integration for s_1 and s_2, and where H is regarded as a function of both s and μ as determined by substituting Eqs. (59), (60), (62), and the change of integration variable to x in Eq. (59). Applying the sort of computations involving Eq. (47) to Eq. (59) we can show that for $|s_1|, |s_2| < L - 6\sqrt{\ln(1/h)}$ the last two integrals in Eq. (63) are bounded in magnitude by h^3 for sufficiently small positive h. Also,

$$|\mu(s_1, x) - \mu(s_2, x)| \leqslant 4(|x| + \max(|s_1|, |s_2|))$$

by the triangle inequality, the established bounds on ϵ, σ, $|\phi|$, and the fact that $|\beta(s, u)| \leqslant |u|$. Using these inequalities, standard results for the moments of normal densities, and the fact that $|s| \leqslant \frac{1}{4} h^{-1/4}$ and $|\phi| \leqslant \frac{4}{3}\sqrt{b}\,h$ allows the expression of inequality (63) to be bounded, after some manipulation, by

$$\frac{1}{4}\sqrt{\ln(1/h)}\ e^{2\max(|s_1|, |s_2|)}\ (h + |s_1 - s_2|)$$

for sufficiently small positive h. Since the indexing of s_1 and s_2 is arbitrary here, this also holds for $s_2 < s_1$. Thus, by the results in Appendix C for Lipschitz conditions for products of functions, and by the Lipschitz conditions for ϕ that can be obtained from those for $\tilde{x}$ and ν,

$$\left|\left[\frac{1}{2}\phi^2 I_2\right](s_1, \rho) - \left[\frac{1}{2}\phi^2 I_2\right](s_2, \rho)\right| \leq \frac{1}{2}\sqrt{\ln(1/h)}\, e^{2\max(|s_1|, |s_2|)}\,(h^3 + h^2|s_1 - s_2|) \tag{64}$$

and

$$\left|\left[\frac{1}{2}\phi^2 I_2\right](s_1, \rho) - \left[\frac{1}{2}\phi^2 I_2\right](s, \rho')\right| \leq \frac{1}{2}\sqrt{\ln(1/h)}\, e^{2|s|}\,(h^3 + h^2|\rho - \rho'|) \tag{65}$$

for sufficiently small positive h, where ρ and ρ' are as in Property 4 of Lemma 2.

Lipschitz Condition for E

From Eq. (36), $E = E_1 + E_2$, where

$$E_1 = \int_{-L}^{L} \alpha\left(\frac{u}{\epsilon}\right)\frac{e^{-\frac{u^2}{2\epsilon^2}}}{\sqrt{2\pi\epsilon^2}}\left[\frac{e^{-\frac{(s-u)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\right] d\lambda_u \tag{66}$$

and

$$E_2 = \phi\int_{-L}^{L} \alpha\left(\frac{u}{\epsilon}\right)\frac{e^{-\frac{u^2}{2\epsilon^2}}}{\sqrt{2\pi\epsilon^2}}\left[\frac{e^{-\frac{(s-u)^2}{2\sigma^2(1+\phi\gamma)^2}}}{\sqrt{2\pi\sigma^2(1+\phi\gamma)^2}}\right]\left\{\left[\frac{s-u}{\sigma(1+\phi\gamma)}\right]^2 - 1\right\} u\, d\lambda_u. \tag{67}$$

Completing the square in the exponent of Eq. (66) gives

$$E_1 = \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\int_{-L}^{L}\frac{e^{-\frac{(u-\epsilon^2 s)^2}{2\sigma^2\epsilon^2}}}{\sqrt{2\pi\sigma^2\epsilon^2}}\,\alpha\left(\frac{u}{\epsilon}\right) d\lambda_u. \tag{68}$$

Denoting the integral in Eq. (68) by $\bar{E}$ gives, for fixed (Z_i, Y_{i-1}),

$$|\bar{E}(s_1) - \bar{E}(s_2)| = \left|\int_{\frac{1}{\sigma}\left(-\frac{L}{\epsilon}-s_1\right)}^{\frac{1}{\sigma}\left(\frac{L}{\epsilon}-s_1\right)}\frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}}\,\alpha(s_1+\sigma\theta)\,d\lambda_\theta - \int_{\frac{1}{\sigma}\left(-\frac{L}{\epsilon}-s_2\right)}^{\frac{1}{\sigma}\left(\frac{L}{\epsilon}-s_2\right)}\frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}}\,\alpha(s_2+\sigma\theta)\,d\lambda_\theta\right|, \tag{69}$$

by a change of the variable of integration in Eq. (68) to $\theta = \frac{1}{\sigma}\left(\frac{u}{\epsilon} - s\right)$ for $s = s_1$ and $s = s_2$. Assuming without loss of generality that $s_1 \leq s_2$ and $|s_1| \leq |s_2|$, the right-hand side of Eq. (69) is bounded by

$$\int_{\frac{1}{\sigma}\left(-\frac{L}{\epsilon}-s_1\right)}^{\frac{1}{\sigma}\left(\frac{L}{\epsilon}-s_2\right)}\frac{e^{-\frac{\sigma^2}{2}}}{\sqrt{2\pi}}\,|\alpha(s_1+\sigma\theta) - \alpha(s_2+\sigma\theta)|\,d\lambda_\theta + \int_{\frac{1}{\sigma}\left(\frac{L}{\epsilon}-s_2\right)}^{\frac{1}{\sigma}\left(\frac{L}{\epsilon}-s_1\right)}\frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}}\,|\alpha(s_1+\sigma\theta)|\,d\lambda_\theta$$

$$+\int_{\frac{1}{\sigma}\left(\frac{L}{\epsilon}-s_2\right)}^{\frac{1}{\sigma}\left|-\frac{L}{\epsilon}-s_1\right|} \frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}} |\alpha(s_2+\sigma\theta)| d\lambda_\theta$$

for $|s_1|, |s_2| < L$, because $0 \leqslant \epsilon \leqslant 1$.

For the case of interest where $|s_1|, |s_2| < L - 6\sqrt{\ln(1/h)}$, the last two integrals in this bound are clearly less than h^3 for sufficiently small positive h because of the magnitude bounds of α. By the Lipschitz conditions assumed for α in Section 4.1, the first integral is bounded by

$$(Ch^3 + Dh^2|s_1 - s_2|) \int_{-\infty}^{\infty} \frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}} e^{2|s_1+\sigma\theta|} d\theta \leqslant (Ch^3 + Dh^2|s_1 - s_2|) e^{2|s_1|} \int_{-\infty}^{\infty} \frac{e^{-\frac{\theta^2}{2}+2|\theta|}}{\sqrt{2\pi}} d\theta$$

$$\leqslant 2e^4 (Ch^3 + Dh^2|s_1 - s_2|) e^{2|s_1|} \int_0^{\infty} \frac{e^{-\frac{(\theta-2)^2}{2}}}{\sqrt{2\pi}} d\theta$$

$$\leqslant 2e^4 (Ch^3 + Dh^2|s_1 - s_2|) e^{2|s_1|}.$$

Combining results gives, for sufficiently small positive h,

$$|\bar{E}(s_1) - \bar{E}(s_2)| \leqslant e^{2\max(|s_1|,|s_2|)} [(2e^4C + 1)h^3 + 2e^4Dh^2|s_1 - s_2|]$$

and, by applying the results of Appendix C,

$$|E_1(s_1) - E_1(s_2)| \leqslant e^{2\max(|s_1|,|s_2|)} [(2e^4C + 1)h^3 + 4e^4Dh^2|s_1 - s_2|], \tag{70}$$

for $|s_1|$ and $|s_2| < L - 6\sqrt{\ln(1/h)}$.

For a fixed s such that $|s| < L - 6\sqrt{\ln(1/h)}$ and ρ and ρ' as in Assumption 4 of Section 4.1, changing the variable of integration in Eq. (68) gives

$$|\bar{E}(\rho) - \bar{E}(\rho')| = \left| \int_{\frac{1}{\sigma}\left(-\frac{L}{\epsilon}-s\right)}^{\frac{1}{\sigma}\left(\frac{L}{\epsilon}-s\right)} \frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}} \alpha(s+\sigma\theta,\rho) d\lambda_\theta - \int_{\frac{1}{\sigma'}\left(-\frac{L}{\epsilon'}-s\right)}^{\frac{1}{\sigma'}\left(\frac{L}{\epsilon'}-s\right)} \frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}} \alpha(s+\sigma'\theta,\rho') d\lambda_\theta \right|, \tag{71}$$

where $\theta = \frac{1}{\sigma}\left(\frac{u}{\epsilon} - s\right)$ in the first integral and $\frac{1}{\sigma'}\left(\frac{u}{\epsilon'} - s\right)$ in the second, and σ' is the value of σ generated by $(Z_i, Y_{i-1})'$ etc. Under these conditions, the regions not common to both integration ranges again contribute a term smaller in magnitude than h^3 for sufficiently small positive h. The rest is bounded by

$$\int_{\theta_1<0}^{\theta_2>0} \frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}} \left|\alpha(s+\sigma\theta,\rho) - \alpha(s+\sigma'\theta,\rho')\right| d\lambda_\theta$$

$$\leqslant \int_{-\infty}^{\infty} \frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}} \left[2Ch^3 + Dh^2(|\rho-\rho'| + |\theta|\,|\sigma-\sigma'|)\right] e^{2|s|} \cdot e^{2|\theta|} d\theta,$$

by the triangle inequality and the Lipschitz conditions assumed for α. From the Lipschitz conditions established in Lemma 1 for $\tilde{x}$ and ν, the results of Appendix C can be used to show that $|\sigma - \sigma'|$ is of order $h|\rho - \rho'|$. This last integral can then be bounded as in the derivation of Eq. (70) by

$$4e^4(Ch^3 + Dh^2|\rho - \rho'|)e^{2|s|}$$

for sufficiently small positive h, so

$$|\bar{E}(\rho) - \bar{E}(\rho')| \leqslant \left[(4e^4C + 1)h^3 + 4e^4Dh^2\,|\,\rho - \rho'|\right]e^{2|s|}$$

and

$$|E_1(s,\rho) - E_1(s,\rho')| < \left[(4e^4C + 1)h^3 + 5e^4Dh^2\,|\,\rho - \rho'|\right]e^{2|s|}. \tag{72}$$

Since $\left|\,\Phi\gamma\left(\frac{u}{\epsilon}\right)\,\right| < \sqrt{h}$ for sufficiently small positive h, it follows from Eq. (67) that

$$|E_2| \leqslant |\Phi|\int_{-L}^{L}\left|\alpha\left(\frac{u}{\epsilon}\right)\right|\frac{e^{-\frac{u^2}{2\epsilon^2}}}{\sqrt{2\pi\epsilon^2}}\left[\frac{e^{-\frac{(s-u)^2}{2\sigma^2(1+\sqrt{h})^2}}}{\sqrt{2\pi\sigma^2(1-\sqrt{h})^2}}\right]\left\{\left[\frac{s-u}{\sigma(1-\sqrt{h})}\right]^2 + 1\right\}|u|\,d\lambda_u. \tag{73}$$

Since $0 \leqslant \sigma, \epsilon \leqslant 1$, completing the square in the exponent gives the bound

$$|E_2| < |\phi|\frac{\sqrt{1+\sqrt{h}}}{1-\sqrt{h}}\frac{e^{-\frac{s^2}{2}+\frac{1}{2}s^2\sqrt{h}}}{\sqrt{2\pi}}\int_{-L}^{L}\frac{e^{\frac{-(u-\bar{u})^2}{2\nu\epsilon^2}}}{\sqrt{2\pi\nu\epsilon 2}}\left|\alpha\left(\frac{u}{\epsilon}\right)\right|\left\{\left[\frac{s-u}{\sigma(1-\sqrt{h})}\right]^2+1\right\}|u|\lambda_u, \tag{74}$$

where

$$\bar{u} = \frac{\epsilon^2 s}{1+\sigma^2\sqrt{h}}$$

and

$$\nu = \frac{\sigma^2(1+\sqrt{h})}{1+\sigma^2\sqrt{h}}.$$

Letting $\theta = \frac{u}{\epsilon}$ and $\bar{\theta} = \frac{\bar{u}}{\epsilon}$ and changing the variable of integration in inequality (74) gives

$$|E_2| < |\phi|\frac{\sqrt{1+\sqrt{h}}}{1-\sqrt{h}}\frac{e^{-\frac{s^2}{2}+\frac{s^2}{2}\sqrt{h}}}{\sqrt{2\pi}}\int_{-\frac{L}{\epsilon}}^{\frac{L}{\epsilon}}\frac{e^{-\frac{(\theta-\bar{\theta})^2}{2\nu}}}{\sqrt{2\pi\nu}}|\alpha(\theta)|\left\{\left[\frac{s-\epsilon\theta}{\sigma(1-\sqrt{h})}\right]^2+1\right\}\epsilon|\theta|\,d\lambda_\theta.$$

Since $\epsilon \leqslant 1$ and since $|\alpha(\theta)| < \Omega h^2 e^{|\theta|}$ in this range for $|s| < L$,

$$s^2 < \frac{1}{16\sqrt{h}}$$

and

$$|E_2| < |\phi|\frac{\sqrt{1+\sqrt{h}}}{1-\sqrt{h}}\left(\frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\right)e^{\frac{1}{32}}\,\Omega h^2\int_{-\infty}^{\infty}\frac{e^{-\frac{(\theta-\bar{\theta})^2}{2\nu}}}{\sqrt{2\pi\nu}}e^{|\theta|}(1+\sqrt{h})\left[\left(\frac{s-\epsilon\theta}{\sigma}\right)^2+1\right]|\theta|\,d\theta. \tag{75}$$

Changing the variable of integration in Eq. (75) to

$$\tau = \frac{\theta - \bar{\theta}}{\sqrt{\nu}},$$

we get

$$|E_2| < |\phi|\Omega\left[\frac{(1+\sqrt{h})^{3/2}}{1-\sqrt{h}}\right]e^{\frac{1}{32}}h^2\frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\exp\left\{-\frac{1}{2}\tau^2+\left|\epsilon s+\sigma\left[\frac{1+\sqrt{h}}{1+\sigma^2\sqrt{h}}\right]\tau\right|\right\}}{\sqrt{2\pi}}$$

$$\times\left\{1+\left[\sigma s+\epsilon\left(\frac{1+\sqrt{h}}{1+\sigma^2\sqrt{h}}\right)\tau\right]^2\right\}\left|\epsilon s+\sigma\left(\frac{1+\sqrt{h}}{1+\sigma^2\sqrt{h}}\right)\tau\right|d\tau. \tag{76}$$

By use of the fact that $\sigma^2 + \epsilon^2 = 1$, the magnitude of the polynomial factor in the integral of Eq. (76) can be bounded by

$$\frac{1}{2}|s|^3 + 2s^2|\tau| + \frac{5}{2}|s|\tau^2 + |s| + 2|\tau|$$

and the nonquadratic exponential term by $|s| + 2|\tau|$ for sufficiently small positive h, so the integral is bounded by

$$e^{|s|}\int_0^{\infty}\frac{e^{-\frac{1}{2}(\tau-1)^2}}{\sqrt{2\pi}}\left(|s|^3 + 4s^2\tau + 5|s|\tau^2 + 2|s| + 4\tau\right)d\tau$$

$$< e^{|s|}\left[|s|^3 + 7|s| + 4(s^2+1)\sqrt{\frac{2}{\pi}}\right] < e^{|s|}(|s|+2)^3$$

by standard results for normal moments. In this case, another bound for this integral is $3e^{2|s|}$, by inequality (E1) of Appendix E. Hence, for $|s| < L$ and sufficiently small positive h,

$$|E_2| < 4\Omega\,|\phi|h^2\frac{e^{-\frac{s^2}{2}+2|s|}}{\sqrt{2\pi}}$$

and

$$|E_2| < 6\Omega\,h^3\left(\frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\right)e^{2|s|}. \tag{77}$$

Lemma 5: In the context of Eqs. (21) to (23), there exists an $h^* > 0$ depending only on the parameters a and b of the unperturbed problem, such that if $h \leqslant h^*$,

$$|\bar{\alpha}(\theta,\rho) - \bar{\alpha}(\theta,\rho')| < e^{2|\theta|}(\bar{C}h^3 + \bar{D}h^2|\rho-\rho'|)$$

and

$$|\bar{\alpha}(\theta,\rho) - \bar{\alpha}(\theta',\rho)| < e^{2\max(|\theta|,|\theta'|)}\,(\bar{C}h^3 + \bar{D}h^2|\theta-\theta'|)$$

for all θ and θ' with magnitude $\leqslant \sqrt{\frac{f_i^2 v}{m_{i+1}}}M - 6\sqrt{\ln\left[\frac{1}{h}\right]}$, where $\bar{\alpha}$ is as in Lemma 3 and ρ and ρ' are as in Assumption 4 of Section 4.1, and where

$$\bar{C} = 6\Omega + 4 + 4e^4C + 3/4\sqrt{\ln(1/h)}$$

and

$$\bar{D} = \frac{3}{4}\sqrt{\ln(1/h)} + L + 5e^4D.$$

Proof: From Eq. (27), $\sqrt{2\pi}\, e^{\frac{s^2}{2}}\, p(s, \bar{L}) < \frac{1}{2} h^3 e^{2|s|}$ under these conditions. Since $p(s) = p(s, L) + p(s, \bar{L})$, the lemma follows from the definition of $\bar{\alpha}$ in Lemma 3 and Eq. (33) and the application of the triangle inequality to the results of Eqs. (41), (56), (57), (64), (65), (70), (72), and (77), because the condition of sufficiently small positive h was only used a finite number of times in these derivations, each time in a way that depended only on the parameters a and b. □

Because of the way the random variable s is defined in terms of the random variable y of Eq. (21), Lemmas 3 to 5 establish the equivalent of Assumptions 1, 2, and 4 of Section 4.1 for y, with the counterparts of $\tilde{x}$, v, k, C, D, and M being $\tilde{x}$, $\frac{m_{i+1}}{f_i^2}$, $k+1$, $\bar{C}$, $\bar{D}$ (as defined in Lemma 5), and

$$M\sqrt{\frac{f_i^2 v}{m_{i+1}}} - 6\sqrt{\ln\left[\frac{1}{h}\right]}.$$

4.4 Measurement Update at Epoch $i+1$

In this subsection, the unsubscripted variable x refers to x_{i+1} instead of x_i, η refers to η^*_{i+1}, and the subscripts are also deleted for m_{i+1}, r_{i+1}, x^*_{i+1}, $\tilde{x}_{i+1}$, v_{i+1}, and n_{i+1}. It is assumed that Assumptions 1, 2, and 3 of Section 4.1 hold at epoch $i+1$ conditioned only on $(Z_i, y_{i-1} * u)$, with $\tilde{x}$, v, k, and (Z_i, Y_{i-1}) replaced by x^*, m, $k+1$, and $(Z_i, Y_{i-1} * u)$ in the notation everywhere. In this context $z = x + n_{i+1}$, so defining

$$t = \frac{x - x^*}{\sqrt{m + r}}, \tag{78}$$

$$\tau = \frac{n_{i+1}}{\sqrt{r}}, \tag{79}$$

$$\sigma = \sqrt{\frac{r}{m + r}}, \tag{80}$$

$$\epsilon = \sqrt{\frac{m}{m + r}}, \tag{81}$$

$$g = \sigma\, \tau, \tag{82}$$

and

$$s = t + g \tag{83}$$

implies that

$$\sigma^2 + \epsilon^2 = 1, \tag{84}$$

$$p(\tau) = \frac{e^{-\frac{1}{2}\tau^2}}{\sqrt{2\pi}}, \tag{85}$$

$$s = \frac{z - x^*}{\sqrt{m + r}}, \tag{86}$$

$$p(s/t) = \frac{e^{-\frac{(s-t)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}, \tag{87}$$

and

$$p(t) = \frac{e^{-\frac{t^2}{2\epsilon^2}}}{\sqrt{2\pi\epsilon^2}}\left[1 + \frac{\eta}{3\epsilon^3}(t^3 - 3\epsilon^2 t) + \alpha\left(\frac{t}{\epsilon}\right)\right]. \tag{88}$$

..ie last for $|t| \leqslant \epsilon M$ in the current notation. It follows from Eq. (86) that conditioning on s (in addition to Z_i) is equivalent to conditioning on z. Hence, if

$$\theta = \frac{t - \bar{k}}{\sqrt{\nu}} \tag{89}$$

for any choice of parameters $\bar{t}$ and ν, then

$$p(\theta/s) = \frac{\sqrt{\nu}\, p_{s/t}(s, \bar{t} + \theta\sqrt{\nu})\, p_t(\bar{t} + \theta\sqrt{\nu})}{p(s)}, \tag{90}$$

by the Bayes rule whenever these densities are defined. If, for a given value of s (or z), the parameter values

$$\bar{t} = \epsilon^2[s + \sigma^2\epsilon\, \eta(s^2 - 1)] \tag{91}$$

and

$$\nu = \sigma^2\epsilon^2(1 + \sigma^2\epsilon\eta s)^2 \tag{92}$$

are selected, then

$$\bar{t} + \theta\sqrt{\nu} = \epsilon(\epsilon s + \sigma\omega), \tag{93}$$

where

$$\omega = \epsilon(\epsilon s + \sigma\theta). \tag{94}$$

Also, the results of Section 4.3 can be specialized to the case of Eqs. (78), (82), (83), (85), and (88) to give

$$p(s) = \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\left[1 + \frac{1}{3}\,\eta\epsilon^3(s^3 - 3s) + \beta(s)\right] \tag{95}$$

for $|s| < \epsilon M - 6\sqrt{\ln(1/h)}$, for sufficiently small positive h, with

$$|\beta| \leqslant 3\Omega\, h^2 e^{|s|} \quad \text{(since } I_2 = 0 \text{ now)} \tag{96}$$

in the context of the current notation. Substituting Eqs. (88), (92), (93), and (95) in Eq. (90), we obtain

$$p(\theta/s) = \frac{e^{-\frac{\omega^2}{2}}\left\{1 + \frac{\eta}{3}[(\epsilon s + \sigma\omega)^3 - 3(\epsilon s + \sigma\omega)] + \alpha(\epsilon s + \sigma\omega)\right\}(1 + \sigma^2\epsilon\eta s)}{\sqrt{2\pi}\left[1 + \frac{1}{3}\eta\epsilon^3(s^3 - 3s) + \beta(s)\right]} \tag{97}$$

if $|s| \leqslant \epsilon M - 6\sqrt{\ln(1/h)}$ and $|t| = |\bar{t} + \sqrt{\nu}\,\theta| \leqslant \epsilon M$. Since $\bar{t}$ and ν are specified from s by Eqs. (91) and (92) now, it follows from the triangle inequality and limits assumed for $|\eta|$ that these two conditions are met if

$$|s| \leqslant \sqrt{8\ln(1/h)} \tag{98}$$

and

$$|\theta| \leqslant M - 3\sqrt{\ln(1/h)} \tag{99}$$

for sufficiently small positive h. From the definitions of s and G_i, it is clear that condition (98) is already implicit in the assumptions here.

Inituitively, it is helpful to note at this point that s is basically the normalized "innovation" variable for the updating step of a Kalman-Bucy filter and θ is the normalized error in the resulting estimate of x.

Now let f denote the function

$$f(\zeta) = \frac{1 + \sigma s \zeta}{\sqrt{2\pi}} \, e^{-\frac{1}{2}[(1 + \sigma s \zeta)\theta + \epsilon(s^2 - 1)\zeta]^2} \tag{100}$$

Substituting Eq. (100) in Eq. (97) and rearranging terms, we get

$$p(\theta/s) = f(\sigma\epsilon\eta)\left\{\frac{1}{1 + \frac{1}{3}\eta\epsilon^3(s^3 - 3s) + \tilde{\beta}(s)\,e^{|s|}} + \frac{\frac{1}{3}\eta[(\epsilon s + \sigma\omega)^3 - 3(\epsilon s + \sigma\omega)]}{1 + \frac{1}{3}\eta\epsilon^3(s^3 - 3s) + \tilde{\beta}(s)\,e^{|s|}}\right\}$$

$$+ \frac{1 + \sigma^2\epsilon\eta s}{\sqrt{2\pi}}\,\tilde{\alpha}(\epsilon s + \sigma\omega)\left[\frac{e^{-\frac{\omega^2}{2} + |\epsilon s + \sigma\omega|}}{1 + \frac{1}{3}\eta(s^3 - 3s) + \tilde{\beta}(s)e^{|s|}}\right] \tag{101}$$

for conditions (98) and (99) and sufficiently small positive h, with

$$|\tilde{\alpha}(\epsilon s + \sigma\omega)| < \Omega h^2$$

and

$$|\tilde{\beta}(s)| < 3\Omega h^2.$$

Also, f and its first two derivatives are continuous for all ζ, so Taylor's theorem with remainder can be applied to f at $\zeta = 0$ to give

$$f(\sigma\epsilon\eta) = \frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}}\,\{1 - \sigma\epsilon\eta[\sigma s(\theta^2 - 1) + \epsilon(s^2 - 1)\theta]\} + \frac{\sigma^2\epsilon^2\eta^2}{2}\,f''(\sigma\epsilon\bar{\eta}), \tag{102}$$

$$\bar{\eta} \in [0, \eta] \cup [\eta, 0],$$

by the use of $\sigma\epsilon\eta$ to play the role of ζ and substitution from Eq. (94) for ω. Using Eq. (102) in Eq. (101) and substituting the expression for f'', we get

$$p(\theta/s) = \frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}}\left(1 + \frac{\sigma^3\eta}{3}(\theta^3 - 3\theta) + \{1 - \sigma\epsilon\eta[\sigma s(\theta^2 - 1) + \epsilon(s^2 - 1)\theta]\}\right.$$

$$\left.\times\,[E_1 + E_2 + \tilde{\beta}(s)e^{|s|}] + T_1 + T_2\right) + \left[\frac{e^{-\frac{1}{2}(\theta + \sigma\epsilon y\bar{\eta})^2}}{\sqrt{2\pi}}\right]\left(\frac{\sigma^2\epsilon^2\eta^2 y}{2}\right)$$

$$\times\left\{1 - \frac{\eta}{3}[\epsilon^3(s^3 - 3s) - (\epsilon s + \sigma\omega)^3 + 3(\epsilon s - \sigma\omega)] + E_1 + E_2 - \tilde{\beta}(s)\,e^{|s|}\right\}$$

$$\times\{[(\theta + \sigma\epsilon y\bar{\eta})^2 - 1](1 + \sigma^2\epsilon s\bar{\eta})y - 2\sigma s(\theta + \sigma\epsilon y\bar{\eta})\} + (1 + E_3)E_4\frac{e^{-\frac{\omega^2}{2} + |\epsilon s + \sigma\omega|}}{\sqrt{2\pi}}, \tag{103}$$

where

$$y(\theta) = \sigma s\theta + \epsilon(s^2 - 1)$$

$$E_1 = \frac{\left[\frac{\eta}{3}\epsilon^3(s^3 - 3s) + \tilde{\beta}(s)e^{|s|}\right]^2}{1 + \frac{1}{3}\eta\epsilon^3(s^3 - 3s) + \tilde{\beta}(s)e^{|s|}},$$

$$E_2 = \frac{\eta}{3}\left[\frac{\frac{1}{3}\eta\epsilon^3(s^3 - 3s) + \tilde{\beta}(s)e^{|s|}}{1 + \frac{1}{3}\eta\epsilon^3(s^3 - 3s) + \tilde{\beta}(s)e^{|s|}}\right][3(\epsilon s + \sigma\omega) - (\epsilon s + \sigma\omega)^3],$$

$$E_3 = \frac{\sigma^2\epsilon\eta s - \frac{1}{3}\eta\epsilon^3(s^3 - 3s) - \tilde{\beta}(s)e^{|s|}}{1 + \frac{1}{3}\eta\epsilon^3(s^3 - 3s) + \tilde{\beta}(s)e^{|s|}},$$

$$E_4 = \tilde{\alpha}(\epsilon s + \sigma\omega),$$

$$T_1 = \frac{\eta}{3}[(\epsilon s + \sigma\omega)^3 - (\epsilon s + \sigma\theta)^3 - 3(\epsilon s + \sigma\omega) + 3(\epsilon s + \sigma\theta)],$$

and

$$T_2 = \frac{\sigma\epsilon\eta^2}{3}[\sigma s(\theta^2 - 1) + \epsilon(s^2 - 1)\theta][\epsilon^3(s^3 - 3s) - (\epsilon s + \sigma\omega)^3 + 3(\epsilon s + \sigma\omega)].$$

Under conditions (98) and (99), it follows from the triangle inequality and Eqs. (84) and (94) that

$$|\epsilon s + \sigma\omega| < M < \frac{1}{4}h^{-1/4}$$

and

$$|\epsilon s + \sigma\omega| < \epsilon\sqrt{1/2\ \ln(1/h)} + \sigma|\theta|$$

for sufficiently small positive h. Using these inequalities and the fact that, for $\zeta > -1$,

$$\left|\frac{1}{1+\zeta} - (1-\zeta)\right| = \frac{\zeta^2}{|1+\zeta|} \in \left[0, \frac{\zeta^2}{1 - |\zeta|}\right]$$

and

$$\left|\frac{1}{1+\zeta} - 1\right| = \left|\frac{\zeta}{1+\zeta}\right| \in \left[0, \frac{|\zeta|}{1 - |\zeta|}\right],$$

and that σ and $\epsilon < 1$, and $\sigma\epsilon < \sqrt{2}$, leads to the following sequence of conclusions for sufficiently small positive h and condition (99):

$$|E_1| < 1000\,h^2 b\ \ln^3(1/h),$$

$$|E_2| < 1000\,h^2 b\ \ln^3(1/h),$$

$$|E_3| < 3h[1 + 8\ln(1/h)]\sqrt{8\ln(1/h)},$$

$$|E_4| = |\tilde{\alpha}| < \Omega h^2,$$

$$|E_1 + E_2 - \tilde{\beta}(s)e^{|s|}| < 2\sigma\Omega h^2 e^{\sqrt{8\ln(1/h)}},$$

$$|y| < 12\ln(1/h)(|\theta| + 1),$$

$$|y\sigma\epsilon\bar{\eta}| < \sqrt{h},$$

$$(\theta + y\sigma\epsilon\bar{\eta}) > \theta^2 - 3\sqrt{h} > \theta^2 - |\theta|,$$

$$|\theta + y\sigma\epsilon\bar{\eta}| < |\theta| + \sqrt{h} < |\theta| + 1,$$
$$(1 + \sigma^2\epsilon s\eta) < 1 + 6h\sqrt{b\ln(1/h)} < 2,$$
$$|1 + \sigma^2\epsilon s\bar{\eta}| < 1 + 6h\sqrt{b\ln(1/h)} < 2,$$
$$T_1 < 3h^2 b\ (|s| + |\theta|)^5 < h^2 b[|\theta| + 2\sqrt{2\ln(1/h)}]^6,$$

and

$$T_2 < 45h^2 b(|s| + |\theta|)^4\ |s||\theta| < 45h^2 b\ [|\theta| + 2\sqrt{2\ln(1/h)}]^6.$$

Also, the last term in Eq. (103),

$$(1 + E_3)\ E_4 \frac{e^{-\frac{\omega^2}{2}+|\epsilon s+\sigma\omega|}}{\sqrt{2\pi}},$$

is bounded in magnitude by $2\Omega\, h^2 e^{-\frac{\omega^2}{2}}\ e^{|\theta|}\ e^{\sqrt{12\ln(1/h)}}$ for sufficiently small positive h under these conditions. But

$$e^{-\frac{\omega^2}{2}} < e^{-\frac{\theta^2}{2}+h^{1/4}}$$

so another bound is $e^{\sqrt{13\ln(1/h)}}\ \Omega\, h^2\ e^{-\frac{\theta^2}{2}+|\theta|}$. These inequalities and the fact that polynomials are ultimately bounded by exponentials lead to the following conclusion, after some manipulation.

Lemma 6: In the context of the notation of this subsection and the assumptions described at its beginning, there exists an $h^* > 0$ depending only on the parameters a and b of the unperturbed problem, such that if $h \leqslant h^*$,

$$\left|\frac{x - \bar{x}}{\sqrt{\nu}}\right| < M - 3\sqrt{\ln(1/h)},$$

and

$$\left|\frac{z - x^*}{\sqrt{m + r}}\right| < \sqrt{8\ln(1/h)},$$

then $p_{\frac{x-\bar{x}}{\sqrt{\nu}}/Z_{i+1}}(\theta, z)$ exists and

$$|\alpha^*| = \left|p_{\frac{x-\bar{x}}{\sqrt{\nu}}/Z_{i+1}}(\theta, z) - \frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}}\left[1 + \frac{\eta\sigma^2}{3}(\theta^3 - 3\theta)\right]\right| < e^{\sqrt{14\ln(1/h)}}\ (\Omega + 1)\ h^2 \frac{e^{-\frac{\theta^2}{2}+|\theta|}}{\sqrt{2\pi}}.$$

This form of the lemma follows by the construction of θ in Eq. (89) and from the fact that the choices of $\bar{t}$ and ν as in Eqs. (91) and (92) make the variables here correspond to those defined by Equation System I.

Lemma 7: In the context of the notation and assumptions described at the beginning of this subsection, if it is also true that Assumption 4 of Section 4.1 holds for ζ and ζ', any two values of a single component of $(Z_i,\ Y_{i-1} * u_i)$ and $(Z_i,\ Y_{i-1} * u_i)'$, then there exists an $h^* > 0$ such that if $h \leqslant h^*$,

$$\left|\frac{x-\tilde{x}}{\sqrt{\nu}}\right| < M - 3\sqrt{\ln(1/h)},$$

and

$$\left|\frac{z-x^*}{\sqrt{m+r}}\right| < \sqrt{8\ln(1/h)},$$

then

$$|\alpha^*(\theta,\zeta) - \alpha^*(\theta',\zeta)| < e^{2\max(|\theta|,|\theta'|)}\,(\tilde{C}h^3 + \tilde{D}h^2|\theta-\theta'|)$$
$$\text{for } |\theta'| < M - 3\sqrt{\ln(1/h)}$$

and

$$|\alpha^*(\theta,\zeta) - \alpha^*(\theta,\zeta')| < e^{2|\theta|}\,(\tilde{C}h^3 + \tilde{D}h^2|\zeta-\zeta'|),$$

where α^* is as defined in Lemma 6 and

$$\tilde{C} = e^{\sqrt{14\ln(1/h)}}\,(C+1)$$

and

$$\tilde{D} = e^{\sqrt{14\ln(1/h)}}\,(D+1).$$

Proof: The results of Section 4.3 can be specialized to the context of Eq. (95) to give

$$|\beta(s,\zeta) - \beta(s,\zeta')| \leqslant 2^{2|s|}\,(\bar{C}h^3 + \bar{D}h^2|\zeta-\zeta'|)$$

and

$$|\beta(s_1,\zeta) - \beta(s_2,\zeta)| \leqslant e^{2\max(|s_1|,|s_2|)}\,(\bar{C}h^3 + \bar{D}h^2|s_1-s_2|)$$

for

$$\bar{C} = 4e^4C + 1 + \frac{1}{2}\sqrt{\ln(1/h)}$$

and

$$\bar{D} = 5e^4D + 1 + \frac{1}{2}\sqrt{\ln(1/h)}.$$

From the Lipschitz conditions already assumed for α and the fact that

$$\tilde{\alpha}(\xi) = \alpha(\xi)\,e^{-|\xi|}$$

and

$$\tilde{\beta}(\xi) = \beta(\xi)\,e^{-|\xi|},$$

it follows from the results of Appendix C that

$$|\tilde{\alpha}(\xi,\zeta) - \tilde{\alpha}(\xi,\zeta')| \leqslant e^{|\xi|}\,(Ch^3 + Dh^2|\zeta-\zeta'|)$$
$$|\tilde{\alpha}(\xi,\zeta) - \tilde{\alpha}(\xi',\zeta)| \leqslant e^{\max(|\xi|,|\xi'|)}\,[Ch^3 + (D+\Omega)\,h^2\,|\xi-\xi'|]$$
$$|\tilde{\beta}(\xi,\zeta) - \tilde{\beta}(\xi,\zeta')| \leqslant e^{|\xi|}\,(\bar{C}h^3 + \bar{D}h^2|\zeta-\zeta'|)$$
$$|\tilde{\beta}(\xi,\zeta) - \tilde{\beta}(\xi',\zeta)| \leqslant e^{\max(|\xi|,|\xi'|)}\,[\bar{C}h^3 + (\bar{D}+3\Omega)\,h^2|\xi-\xi'|]$$

for the conditions of interest here. The lemma is then established by repeated application of the results of Appendix C to the composite expression of Eq. (103) for $p(\theta/s)$, using the definition of θ and s in terms of the variables of interest, various inequalities developed earlier in this subsection, and the fact that

$$|(\epsilon s + \sigma\omega) - (\epsilon s + \sigma\theta)| < h\sqrt{\ln(1/h)},$$

$$\frac{\partial}{\partial\zeta}\,[\epsilon s + \sigma\omega(\theta)] = \theta \times 0(h),$$

and

$$\max_{\zeta>0}\,\{\zeta^p e^{-\zeta}\} = \left(\frac{p}{e}\right)^p \quad \text{for } p \geqslant 1. \qquad \square$$

To establish probability bounds for extreme values of the estimate error, suppose first that y is a value such that

$$y > z - x^* > 0,$$

and for small $\Delta > 0$ consider

$$\Pr\{x - x^* > y/x + n \in [z,\ z + \Delta)\}.$$

From the definition of conditional probability, this probability is bounded by

$$\frac{\Pr\{x - x^* > y \ \text{ and } \ x + n \in [z,\ z + \Delta)\}}{\Pr\{x + n \in [z,\ z + \Delta) \ \text{ and } \ 2z + \Delta - y \leqslant x - x^* \leqslant y\}}.$$

Since x and n are independent (given only Z_i and Y_i), and since n is normal $(0, r)$, the numerator of this last expression is bounded above by

$$\Delta\,\frac{e^{-\frac{(x^*+y-z-\Delta)^2}{2r}}}{\sqrt{2\pi r}}\int_{x^*+y}^{\infty} dP(x)$$

and the denominator is bounded below by

$$\Delta\,\frac{e^{-\frac{(x^*+y-z)^2}{2r}}}{\sqrt{2\pi}}\int_{2z+\Delta-(x^*+y)}^{x^*+y} dP(x).$$

Hence,

$$\Pr\{x - x^* > y\,/\,(x+n) \in [z,\ z+\Delta]\} \leqslant \frac{e^{-\frac{\Delta(y-z+x^*)}{r}}\ \Pr\{x-x^*>y\}}{1-\Pr\{x-x^*>y\}-\Pr\{x-x^*<2(z-x^*)-y+\Delta\}}.$$

Taking the limit as $\Delta \to 0$ gives

$$\Pr\{x - x^* > y/z\} \leqslant \frac{\Pr\{x - x^* > y\}}{1-\Pr\{x-x^*>y\}-\Pr\{x-x^*<2(z-x^*)-y\}} \tag{104}$$

For $\Pr\{x - x^* < -y/z\}$, this construction shows that the inequality is at least as great as (104). Similarly, for $y < z - x^* < 0$,

$$\Pr\{x - x^* < y/z\} \leqslant \frac{\Pr\{x - x^* > y\}}{1-\Pr\{x-x^*>y\}-\Pr\{x-x^*>2(z-x^*)-y\}}$$

and there is at least as great an inequality for $\Pr\{x - x^* > -y/z\}$. Combining these results for $y > 3|z - x^*|$ gives

$$\Pr\{|x-x^*| > y/z\} \leqslant \frac{\Pr\{|x-x^*|>y\}}{1-\Pr\left\{|x-x^*|>\frac{1}{3}y\right\}}, \tag{105}$$

where all probabilities are also conditioned on Z_i (not shown in the notation).

From Equation System I,

$$x - x^* = (x - \hat{x}) + (\hat{x} - x^*) = \theta\sqrt{v} + \frac{ms}{\sqrt{m+r}} + \frac{r\eta m^{3/2}}{(m+r)^2}\ (s^2 - 1),$$

where, as before in this subsection,

$$\eta = \eta^*_{i+1},$$

$$s = \frac{z - x^*}{\sqrt{m+r}},$$

and

$$\theta = \frac{x - \tilde{x}}{\sqrt{v}}.$$

For $y \geqslant 0$, therefore,

$$|\theta| \geqslant y \Rightarrow |x - x^*| \geqslant y\sqrt{v} - \sqrt{m}\,|s + \eta(s^2 - 1)|$$

$$\Rightarrow |x - x^*| \geqslant y\sqrt{a} - 3\sqrt{b \ln (1/h)},$$

for sufficiently small positive h, because of the bounds assumed for $|m|$, $|\eta|$, and $|s|$. Hence, for sufficiently small positive h and $y > \sqrt{\frac{b}{a} \ln (1/h)}$,

$$|\theta| \geqslant y \Rightarrow |x - x^*| \geqslant \frac{1}{2}\, y\sqrt{a}. \tag{106}$$

By assumption, $|z - x^*| < \sqrt{8b \ln (1/h)}$, so for sufficiently small positive h,

$$y > \frac{1}{8} h^{-\frac{1}{8}} \Rightarrow \frac{y\sqrt{a}}{2} > 3|z - x^*|.$$

In this case, inequality (105) can be applied to (106) to give

$$\Pr\{|\theta| \geqslant y/z\} \leqslant \left[\frac{k+1}{1 - (k+1)\, e^{-\frac{1}{2}\left(\frac{y^2 a}{6b}\right)^{\frac{1}{k+1}}}}\right] e^{-\frac{1}{2}\left(\frac{y^2 a}{2b}\right)^{\frac{1}{k+1}}} \tag{107}$$

by use of the inequality established in Lemma 4. Furthermore, for sufficiently small positive h,

$$\left.\begin{array}{l} i < \frac{1}{2}\sqrt{\ln (1/h)} \\ y > \frac{1}{8} h^{-1/8} \end{array}\right\} \Rightarrow \ln y > -\ln 8 + \frac{1}{8} \ln(1/h) > (2i + 2) \ln \left(\frac{2b}{a}\right)$$

$$\Rightarrow y > \left(\frac{2b}{a}\right)^{2i+2}$$

$$\Rightarrow \frac{y^2 a}{2b} > (y^2)^{\frac{2i+2}{2i+3}} \quad \text{(since } k = 2i + 1)$$

$$\Rightarrow e^{-\frac{1}{2}\left(\frac{y^2 a}{2b}\right)^{\frac{1}{k+1}}} \leqslant e^{-\frac{1}{2} y^{\frac{2}{k+2}}}$$

and

$$\left.\begin{array}{l} i < \frac{1}{2}\sqrt{\ln(1/h)} \\ y > \frac{1}{8}h^{-1/8} \end{array}\right\} \Rightarrow \ln y > -\ln 8 + \frac{1}{8}\ln(1/h) > \ln\left(3\frac{\sqrt{b}}{a}\right) + (2i+2)\,[\ln 4 + \ln\ln(2i+2)]$$

$$\Rightarrow y > 3\sqrt{\frac{b}{a}}\,[4\ln(2i+2)]^{2i+2}$$

$$\Rightarrow e^{\frac{1}{2}\left(\frac{y^2 a}{6b}\right)^{\frac{1}{k+1}}} \geqslant (k+1)(k+2)$$

$$\Rightarrow 1 - (k+1)\,\frac{e^{-\frac{1}{2}\left(\frac{y^2 a}{6b}\right)^{\frac{1}{k+1}}}}{k+1} \geqslant 1 - \frac{1}{k+2}$$

$$\Rightarrow \frac{k+1}{1-(k+1)\,e^{-\frac{1}{2}\left(\frac{y^2 a}{6b}\right)^{\frac{1}{k+1}}}} \leqslant k+2.$$

Hence, using Eq. (107) and the definitions of θ and s, we establish the following lemma.

Lemma 8: In the context of the notation and assumptions adopted at the beginning of this subsection, there exists an $h^* > 0$ depending only on the parameters a and b of the unperturbed problem, such that if $h \leqslant h^*$ and the epoch index i is less than $\frac{1}{2}\sqrt{\ln(1/h)}$, then

$$\Pr\left\{\left|\frac{x-\tilde{x}}{\sqrt{v}}\right| > y/Z_{i+1}\right\} < (k+2)\,e^{-\frac{1}{2}y^{\frac{2}{k+2}}} \quad \text{for all } y > \frac{1}{8}h^{-1/8}.$$

In summary, the results of this subsection establish counterparts to Assumptions 1, 2, and 4 of Section 4.1 at epoch $i+1$ if $i < \frac{1}{2}\sqrt{\ln(1/h)}$.

4.5 Implications for a Sequence of Epochs

If an error is multiplied by a factor not exceeding a constant H in magnitude at each epoch, and an additional error not exceeding H in magnitude is added, it can easily be shown by induction on N that the accumulated error after N epochs is less than

$$(H+1)^N$$

if the initial error is zero. Hence, if

$$N \leqslant \frac{1}{2}\sqrt{\frac{1}{\beta}\ln(1/h)}, \quad \beta \text{ a constant},$$

and

$$H \leqslant e^{\sqrt{\beta\ln(1/h)}} - 1,$$

then

$$(H+1)^N \leqslant e^{N\sqrt{\beta \ln(1/h)}} \leqslant e^{\frac{1}{2}\ln(1/h)}$$

$$\leqslant h^{-1/2}$$

$$\leqslant \frac{1}{4}\, h^{-1/4} \text{ for } h < \frac{1}{256}.$$

Theorem 1: Given the restrictions imposed on the unperturbed problem in Section 3.1, if the number of epochs N is such that

$$N \leqslant \frac{1}{8}\sqrt{\ln(1/h)},$$

and if $(Z_i,\ Y_{i-1}) \in G_i$ (where $i \in \{0,\ 1,\ \ldots,\ N\}$), then Assumptions 1, 2, and 4 of Section 4.1 hold for epoch i in the problem with perturbations, if h is smaller than some strictly positive value h^* which depends only on the parameters a, b, and F of the unperturbed problem (and not on the epoch index).

Proof (induction on i): These assumptions are clearly true at epoch 0, with $M_0 = \frac{1}{4}\, h^{-1/4}$, $\Omega_0 = C_0 = D_0 = 0$, and $h^* = 2^{-32}$. Assumption 2 follows for $h < 2^{-32}$, by the inequality for normal tails,

$$\int_\tau^\infty \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\, dt < \frac{e^{-\frac{\tau^2}{2}}}{\tau\sqrt{2\pi}} \text{ for } \tau > 0.$$

The induction step can be established by noting that x_{i+1} can be constructed from x_i as the composition of the two transformations

$$y = x_i + (1 + \psi_i x_i)\, w$$

and

$$x_{i+1} = f_i y + u_i,$$

where $w = \dfrac{w_i}{f_i}$, a normal $\left(0,\ \dfrac{q_i}{f_i^2}\right)$ random variable. Since $(Z_i * z_{i+1},\ Y_{i-1} * u_i) \in G_{i+1}$ $\Rightarrow (Z_i,\ Y_{i-1}) \in G_i$, if it is assumed that Assumptions 1, 2, and 4 of Section 4.1 hold with

$$\Omega_i \leqslant A_i,$$

$$C_i \leqslant A_i,$$

and

$$D_i \leqslant A_i,$$

it follows from Lemmas 2 through 5, applied to these two transformations in turn, that it holds at epoch $i+1$ with conditioning only on Z_i, for corresponding constants

$$\overline{\Omega} = 3A_i + b\left(\frac{1}{F^2} + 6,000,000\right),$$

$$\overline{C} = (6 + 4e^4)A_i + \sqrt{\ln(1/h)},$$

$$\overline{D} = 5e^4 A_i + \sqrt{\ln(1/h)},$$

and

$$\overline{M} = F\sqrt{\frac{a}{b}}\, M_i - 6\sqrt{\ln(1/h)}$$

if h is smaller than some strictly positive quantity that does not depend on i, since

$$i < \frac{1}{8}\sqrt{\ln(1/h)} \Rightarrow i < \frac{1}{2}\, h^{-\frac{1}{12}}$$

for sufficiently small positive h. Also, the parameter $k = 2i + 1$ changes to $2i + 2$ here. If $\overline{M} > \frac{1}{4}\, h^{-\frac{1}{8}}$, Lemmas 6 through 8 imply that the hypothesis also holds at epoch $i + 1$, with conditioning on Z_{i+1}, for

$$M_{i+1} = \overline{M} - 3\sqrt{\ln(1/h)} = F\sqrt{\frac{a}{b}}\, M_i - 9\sqrt{\ln(1/h)},$$

$$\Omega_{i+1} = e^{\sqrt{14\ln(1/h)}}\,(\overline{\Omega} + 1),$$

$$C_{i+1} = e^{\sqrt{14\ln(1/h)}}\,(\overline{C} + 1),$$

and

$$D_{i+1} = e^{\sqrt{14\ln(1/h)}}\,(\overline{D} + 1).$$

Thus, for sufficiently small positive h, a corresponding bound A_{i+1} is

$$A_{i+1} = e^{4\sqrt{\ln(1/h)}}\,(A_i + 1).$$

By the remarks at the beginning of this subsection, therefore, Ω_i, C_i, and D_i do not exceed $\frac{1}{4}\, h^{-1/4}$, for sufficiently small positive h and $i < \frac{1}{8}\sqrt{\ln(1/h)}$, and it is also clear that such a positive upper limit for h exists, so that $M_i > \frac{1}{4}\, h^{-1/8}$ for $M_0 = \frac{1}{4}\, h^{-1/4}$ and $i < \frac{1}{8}\sqrt{\ln(1/h)}$. Since the condition of sufficiently small positive h was used in this proof only a finite number of times, in a way which did not depend on the index i of the induction step, and depended only on the parameters a, b, and F (sometimes via Lemmas 3 through 8), the induction step is verified. □

4.6 Moment Error Bounds

In this subsection it is assumed that $(Z_N, Y_{N-1}) \in G_N$, and the first four central moments of

$$p_{\frac{x_i - \hat{x}_i}{\sqrt{v_i}}/Z_i},$$

if they exist, are denoted by $\bar{x}_i$, σ_i^2, $2\theta_i$, and γ_i respectively for epoch $i \leqslant N$. For a given epoch i, with

$$i \leqslant \frac{1}{8}\sqrt{\ln(1/h)}$$

and

$$\frac{1}{4}\, h^{-1/8} \leqslant M_i \leqslant \frac{1}{4}\, h^{-1/4},$$

we suppress the epoch subscript and the conditioning on Z_i in the notation, and note that $(Z_i, Y_{i-1}) \in G_i$.

Lemma 9: In the current context, there exists an $h^* > 0$ such that for $h \leqslant h^*$ and $n = 1, 2, 3, 4$

$$\int_M^\infty t^n \, dP_{\frac{x-\bar{x}}{\sqrt{v}}}(t) < h^3.$$

Proof: Since $\Pr\left\{\left|\frac{x-\tilde{x}}{\sqrt{v}}\right| > t\right\} < ke^{-\frac{1}{2}t^{2/k}}$, $k = 2i + 1$ for $|t| \geqslant M$, it follows from the construction of the Lebesgue integral and the monotonicity of t^n on $[M, \infty)$ that

$$0 \leqslant \int_M^\infty t^n \, dP_{\frac{x-\bar{x}}{\sqrt{v}}}(t) < k \int_M^\infty t^n e^{-\frac{1}{2}t^{2/k}} \, dt,$$

where $k = 2i + 1 < 1/2\sqrt{\ln(1/h)}$. Changing the integration variable to $u = t^{1/k}$ gives

$$\int_M^\infty t^n \, dP_{\frac{x-\bar{x}}{\sqrt{v}}}(t) < k^2 \int_{M^{1/k}}^\infty u^{(n+1)k-1} e^{-\frac{u^2}{2}} \, du.$$

Integration by parts gives this last bound as

$$k^2 \{M^{n+1-\frac{2}{k}} e^{-\frac{1}{2}M^{2/k}} + [(n+1)k - 2] \int_{M^{1/k}}^\infty u^{(n+1)k-3} e^{-\frac{u^2}{2}} \, du\}.$$

Hence, it is easy to show by induction on k that

$$\int_M^\infty t^n \, dP_{\frac{x-\bar{x}}{\sqrt{v}}}(t) < k^2 (M^{1/k} + 2)^{k(n+1)-2} e^{-\frac{1}{2}M^{2/k}}.$$

For sufficiently small positive h, $k < 1/2\sqrt{\ln(1/h)}$, $M > 1/4h^{-1/8}$, and

$$M^{1/k} > (4h^{1/8})^{-\frac{2}{\sqrt{\ln(1/h)}}} = e^{-\frac{2\ln(4h^{1/8})}{\sqrt{\ln(1/h)}}} = e^{\frac{\frac{1}{4}\ln(1/h) - 2\ln 4}{\sqrt{\ln(1/h)}}} > 2,$$

in which case

$$\int_M^\infty t^n \, dP_{\frac{x-\bar{x}}{\sqrt{v}}}(t) < k^2 \, 2^{k(n+1)} M^{n+1} e^{-\frac{1}{2}M^{2/k}} < k^2 e^{k(n+1)} M^{n+1} e^{-\frac{1}{2}M^{2/k}}.$$

The natural logarithm of this last bound is $2\ln k + k(n+1) + (n+1)\ln M - \frac{1}{2}M^{2/k}$, which is less than

$$2\ln[\sqrt{\ln(1/h)}] + (n+1)[\sqrt{\ln(1/h)} + \frac{1}{2}\ln(1/h)] - \frac{1}{8}\left(\frac{1}{h}\right)^{\frac{1}{4\sqrt{\ln(1/h)}}}$$

under these conditions. For sufficiently small positive h, this expression is dominated by the last (negative) term, which can be rewritten as

$$-\frac{1}{8} e^{\frac{(1/h)}{4\sqrt{\ln(1/h)}}}$$

and can clearly be made less than $-3\ln(1/h)$ by taking h to be a small enough positive value. Since this is the logarithm of the bound of interest,

$$\int_M^\infty t^n \, dP_{\frac{x-\bar{x}}{\sqrt{v}}}(t) < h^3$$

for all $h \leqslant h^*$ for some $h^* > 0$. □

In the case of the first central moment,

$$E(x) = \bar{x} \triangleq \tilde{x} + \sqrt{v}\ E\left(\frac{x-\tilde{x}}{\sqrt{v}}\right),$$

where E denotes expectation (conditioned on Z_i) and $\tilde{x}$ and v are as generated by Equation System I. Thus,

$$\begin{aligned} x - \tilde{x} &= \sqrt{v} \int_{-\infty}^{\infty} t dP_{\frac{x-\tilde{x}}{\sqrt{v}}}(t) \\ &= \sqrt{v}\left[\int_{-\infty}^{\infty} t\bar{p}(t)dt - \int_A t\bar{p}(t)dt + \int_{-M}^{M} t\beta(t)d\lambda_t + \int_A tdP_{\frac{x-\tilde{x}}{\sqrt{v}}}(t)\right], \end{aligned} \tag{108}$$

where

$$\bar{p}(t) = \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\ [1 + \frac{\eta}{3}(t^3 - 3t)] \quad \text{(the approximation)},$$

$$A = (-\infty,\ M] \cup [M,\ \infty),$$

and

$$\beta(t) = p_{\frac{x-\tilde{x}}{\sqrt{v}}}(t) - \bar{p}\ (t) \quad \text{(the error function)}.$$

Also, there are no moment existence problems because of Lemma 9. From the results of earlier subsections,

$$|\beta(t)| < \frac{2\Omega\ h^2\ e^{-\frac{t^2}{4}}}{\sqrt{4\pi}}$$

for $|t| < M$ and sufficiently small positive h. The first integral in Eq. (108) is zero, so from this inequality and that established for $|\eta|$ when $(Z_i,\ Y_{i-1}) \epsilon\ G_i$,

$$\begin{aligned} |\bar{x} - \tilde{x}| \leqslant \sqrt{b}\ \Bigg\{ & 2\int_{\frac{1}{4}h^{-1/8}}^{\infty} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}[t + h\sqrt{b}\ (t^4 + 3t^2)]\ dt \\ & + 4\Omega\, h^L \int_0^{\infty} t\ \frac{e^{-\frac{t^2}{4}}}{\sqrt{4\pi}}\ dt + \int_M^{\infty} t\ dP_{\frac{x-\tilde{x}}{\sqrt{v}}}(t)\Bigg\}. \end{aligned} \tag{109}$$

The first integral in Eq. (109) can be made less than h^3 for sufficiently small positive h by standard results for normal tails, and the second integral is $\pi^{-1/2}$. By Lemma 9, therefore,

$$|\bar{x} - \tilde{x}| < \sqrt{b}\left(\frac{4\Omega}{\sqrt{\pi}} + 3\right) h^2. \tag{110}$$

With respect to the conditional distribution given Z_i,

$$E(x - \bar{x})^2 \triangleq \sigma^2 = (\bar{x} - \tilde{x})^2 + vE\left(\frac{x - \tilde{x}}{\sqrt{v}}\right)^2 \tag{111}$$

and

$$E\left(\frac{x-\tilde{x}}{\sqrt{\nu}}\right)^2 = \int_{-\infty}^{\infty} t^2 \, dP_{\frac{x-\tilde{x}}{\sqrt{\nu}}}(t)$$

$$= \int_{-\infty}^{\infty} t^2 \bar{p}(t)\, dt - \int_A t^2 \bar{p}(t)\, dt + \int_{-M}^{M} t^2 \beta(t)\, dt + \int_A t^2 \, dP_{\frac{x-\tilde{x}}{\sqrt{\nu}}}(t).$$

Using Lemma 9 with $n = 2$ and the same type of reasoning as for the first central moment, we can show that this last expression is

$$1 + \text{error}, \text{ where } |\text{error}| < (4\sqrt{2}\,\Omega + 3)\, h^2.$$

For sufficiently small positive h, $(\bar{x} - \tilde{x})^2$ is of order h^4, and so

$$|\sigma^2 - \nu| < b(4\sqrt{2}\,\Omega + 3)\, h^2. \tag{112}$$

By the preceding results,

$$E(x - \bar{x})^3 \triangleq 2\theta = 3(\tilde{x} - \bar{x})^2 E(x - \tilde{x})^2 - 2(\tilde{x} - \bar{x})^2 + \nu^{3/2} E\left[\left(\frac{x-\tilde{x}}{\sqrt{\nu}}\right)^3\right]$$

$$= \nu^{3/2} E\left[\left(\frac{x-\tilde{x}}{\sqrt{\nu}}\right)^3\right].$$

Repeating the above analysis using t^3 and $n = 3$ we get

$$2\theta = 2\nu^{3/2}\eta + \text{error}, \ |\text{error}| < b^{3/2}\left(16\frac{\Omega}{\sqrt{\pi}} + 3\right),$$

where η is as generated by Equation System I. By the definition of $\bar{\lambda}$, therefore,

$$|\theta - \bar{\lambda}| < \left(\frac{8\Omega}{\sqrt{\pi}} + \frac{3}{2}\right) b^{3/2} h^2 \tag{113}$$

for sufficiently small positive h.

Again expressing $x - \bar{x}$ as $(x - \tilde{x}) + (\tilde{x} - \bar{x})$,

$$E(x - \bar{x})^4 \triangleq \gamma = E[(x - \tilde{x})^4 + 4(\tilde{x} - \bar{x})(x - \tilde{x})^3 + 6(x - \tilde{x})^2(\tilde{x} - \bar{x})^2]$$
$$+ 4(\bar{x} - \tilde{x})(\tilde{x} - \bar{x})^3 + (\bar{x} - \tilde{x})^4. \tag{114}$$

From the preceding results for lower moments and the fact that $|\eta| < 3h\sqrt{b}$ under the present conditions, the only term in Eq. (114) that is significant to order h is

$$E(x - \tilde{x})^4 = \nu^2 \int_{-\infty}^{\infty} t^4 \, dP_{\frac{x-\tilde{x}}{\sqrt{\nu}}}(t)$$

$$= 3\nu^2 + \text{error}, \ |\text{error}| < b^2 h$$

by a similar analysis. So

$$|\gamma - 3\nu^2| < b^2 h \tag{115}$$

for sufficiently small positive h.

Lipschitz Conditions

If for a given epoch l, $l \le N$, $(Z_l, Y_{l-1})' \in G_l$ differs from the initial segment (Z_l, Y_{l-1}) of (Z_N, Y_{N-1}) in only one component, for which the respective values are denoted as ρ' and ρ, it follows from Eq. (108) and the analysis in this subsection that

$$\delta_1(\rho) - \delta_1(\rho') = \sqrt{\nu(\rho)}\left[\epsilon_1 + \int_{-M}^{M} t\beta(t, \rho)\, d\lambda_t\right] - \sqrt{\nu(\rho')}\left[\epsilon_2 + \int_{-M}^{M} t\beta(t, \rho')\, d\lambda_t\right], \qquad (116)$$

where $\delta_1 = \bar{x} - \tilde{x}$ and $\nu(\rho')$ denotes $\nu[(Z_l, Y_{l-1})']$, etc ., with $|\epsilon_1|$ and $|\epsilon_2| < 3h^3$ for sufficiently small positive h. Rearranging terms in Eq. (116) gives

$$\delta_1(\rho) - \delta_1(\rho') = \sqrt{\nu(\rho)}\left\{(\epsilon_1 - \epsilon_2) + \int_{-M}^{M} t[\beta(t,\rho) - \beta(t,\rho')]\, d\lambda_t\right\}$$

$$+ [\bar{x}(\rho') - \tilde{x}(\rho')]\,[\sqrt{\nu(\rho)} - \sqrt{\nu(\rho')}]. \qquad (117)$$

Since

$$\beta(t,\rho) = \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\,\alpha(t,\rho)$$

by definition, it follows from the Lipschitz conditions established by Lemma 1 and Theorem 1 for ν and α (using also the results of Appendix C in the case of ν) and from inequality (110) that

$$|\delta_1(\rho) - \delta_1(\rho')| \le \sqrt{b}\left[6h^3 + 2\int_0^{\infty} t(Ch^3 + Dh^2|\rho - \rho'|)\,\frac{e^{-\frac{t^2}{2}+2t}}{\sqrt{2\pi}}\,dt + \left(\frac{4\Omega}{\sqrt{\pi}} + 3\right)\left(\frac{\Gamma}{2\sqrt{a}}\right) h^3|\rho - \rho'|\right] \qquad (118)$$

for sufficiently small positive h. Performing the integration in (118) then gives, for sufficiently small positive h,

$$|\delta_1(\rho)-\delta_1(\rho')| \le \sqrt{b}\left[6+e^2\left(2+\sqrt{\frac{2}{\pi}}\right)C\right]h^3+\left[\sqrt{b}\,e^2\left(2+\sqrt{\frac{2}{\pi}}\right)D+1\right]h^2|\rho-\rho'|. \qquad (119)$$

Since C and D are less than $h^{-1/4}$,

$$|\delta_1(\rho) - \delta_1(\rho')| < (2h^{-1/4})\,h^3 + (2h^{-1/4})\,h^2|\rho - \rho'| \qquad (120)$$

for sufficiently small positive h.

From Eq. (111) and the fact that $(\bar{x} - \tilde{x})^2 < \frac{1}{2}h^3$ for sufficiently small positive h,

$$\delta_2(\rho) - \delta_2(\rho') = \nu(\rho)\left[\epsilon_3 + \int_{-M}^{M} t^2\beta(t,\rho)\, d\lambda_t\right] - \nu(\rho')\left[\epsilon_4 + \int_{-M}^{M} t^2\beta(t,\rho')\, d\lambda_t\right],$$

with $|\epsilon_3|$, $|\epsilon_4| < h^3$, where $\delta_2 = \sigma^2 - \nu$.

Rearranging terms, using the triangle inequality, and using the magnitude bounds and Lipschitz conditions established for α, we obtain

$$|\delta_2(\rho) - \delta_2(\rho')| \le \nu(\rho)\left[2h^3 + \int_{-M}^{M} t^2\,\frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\,|\alpha(t,\rho) - \alpha(t,\rho')|\,d\lambda_t\right]$$

$$+ \left[h^3 + \int_{-M}^{M} t^2 \frac{e^{-t^2}}{\sqrt{2\pi}} |\alpha(t,\rho)| \, d\lambda_t\right] |\nu(\rho) - \nu(\rho')|$$

$$\leqslant b\left[2h^3 + \int_{-\infty}^{\infty} t^2(Ch^3 + Dh^2|\rho - \rho'|) \frac{e^{-\frac{t^2}{2}+2|t|}}{\sqrt{2\pi}} \, dt\right]$$

$$+ \left[h + \Omega \int_{-\infty}^{\infty} t^2 \frac{e^{-\frac{t^2}{2}+|t|}}{\sqrt{2\pi}} \, dt\right] \Gamma h^3|\rho - \rho'|$$

for sufficiently small positive h. Evaluating the integrals in this last inequality, we get

$$|\delta_2(\rho) - \delta_2(\rho')| \leqslant b\left[2h^3 + e^2\left(5 + 8\sqrt{\frac{2}{\pi}}\right)(Ch^3 + Dh^2|\rho - \rho'|)\right]$$

$$+ \Gamma h^3\left[h + \Omega e\left(2 + 2\sqrt{\frac{2}{\pi}}\right)\right] |\rho - \rho'| \tag{121}$$

in this case. Also, for sufficiently small positive h,

$$|\delta_2(\rho) - \delta_2(\rho')| < (2h^{-1/4})h^3 + (2h^{-1/4})h^2 \, |\rho - \rho'|, \tag{122}$$

because C and D are both less than $h^{-1/4}$.

Since, as before, the condition of sufficiently small positive h was invoked only a finite number of times in their derivation, Eqs. (110), (112), (113), (115), (119), and (121) can be summarized in the following result.

Lemma 10: There exists an $h^* > 0$ such that if $h \leqslant h^*$, and if (Z_i, Y_{i-1}) and $(Z_i, Y_{i-1})' \in G_i$ differ in at most one component with respective values denoted as ρ and ρ' for $i \leqslant N \leqslant \frac{1}{8}\sqrt{ln(1/h)}$, then the first four central moments, $\bar{x}_i$, σ_i^2, $2\theta_i$, and γ_i of

$$\frac{p_{x_i - \hat{x}_i}}{\sqrt{\nu_i}}$$

exist and

$$|\bar{x}_i - \hat{x}_i| < \sqrt{b}\left(\frac{4}{\sqrt{\pi}}\,\Omega_i + 3\right) h^2,$$

$$|\sigma_i^2 - \nu_i| < b(4\sqrt{2}\,\Omega_i + 3)h^2,$$

$$|\theta_i - \bar{\lambda}_i| < b^{3/2}\left(\frac{8}{\sqrt{\pi}}\,\Omega_i + 3\right) h^2,$$

$$|\gamma_i - 3\nu_i^2| < b^2 h,$$

$$|\delta_1(\rho) - \delta_1(\rho')| < \sqrt{b}\left[6 + e^2\left(2 + \sqrt{\frac{2}{\pi}}\right) C_i\right] h^3 + \left[\sqrt{b}e^2\left(2 + \sqrt{\frac{2}{\pi}}\right) D_i + 1\right] h^2 \, |\rho - \rho'|,$$

and

$$|\delta_2(\rho) - \delta_2(\rho')| < b\left[2 + e^2\left(5 + 8\sqrt{\frac{2}{\pi}}\right) C_i\right] h^3 + \left[be^2\left(5 + 8\sqrt{\frac{2}{\pi}}\right) D_i + 1\right] h^2 \, |\rho - \rho'|,$$

where $\tilde{x}_i$, ν_i, and $\bar{\lambda}_i = \nu_i^{3/2}\eta_i$ are as generated by Equation System I and

$$\delta_1 = \bar{x}_i - \tilde{x}_i$$

and

$$\delta_2 = \sigma_i^2 - \nu_i.$$

5. ADEQUACY FOR OPTIMAL CONTROL APPROXIMATION

One common use of state-estimation results is in the implementation of optimal control laws. It is shown in this section that the error bounds established in the previous section are strong enough to guarantee that the control values computed in such a way according to a certain first-order formal analysis (to be described shortly) are indeed within order h of being the optimal values, except perhaps for a set of realizations of small probability, when we minimize the expected value of a quadratic performance criterion with the dynamics of Eq. (1) — u_i being the control variable — and with the state measurements of Eq. (2). Also, the set of realizations for which this control accuracy does not hold is shown to be contained in another set of realizations for which the prior probability goes to zero as h does and it is always known to the controller from available data whenever the current initial segment of a realization is such that the realization cannot be in this set.

5.1 Control Problem Formulation

The particular control problem considered here is that of minimizing the performance criterion

$$J = E\left[\frac{1}{2}\, s_N x_N^2 + \sum_{i=0}^{N-1} \frac{1}{2}(a_i x_i^2 + b_i u_i^2)\right], \tag{123}$$

where E denotes prior expected value and the x_i are determined by Eq. (1), with the u_i now regarded as control variables. Also $s_N \geqslant 0$ and, for $i = 0, \ldots, N-1$, $A \geqslant a_i \geqslant 0$ and $b_i \geqslant B > 0$. The control variables are generated from the available data by a Borel-measurable control law U, i.e.,

$$U\colon (Z_i, Y_{i-1}) \rightarrow u_i,\ i = 0, \ldots, N-1. \tag{124}$$

Thus, the convention used here is that the controller receives the measurement z_i at each epoch i before having to decide the value of the control u_i. Also, we let ξ denote the set of *admissible* control laws, i.e., those having the functional dependence indicated by expression (124) such that these functions are Borel measurable (so that all quantities determined thereby are well-defined random variables).

A particular control law $U_0 \in \xi$ is called *optimal* if and only if

$$J(U_0) = \inf\,\{J(U) : U \in \xi\}.$$

This infimum clearly exists because J is necessarily nonnegative by its definition and J is obviously finite under the trivial (and admissible) control law which always gives $u_i = 0$. For a given epoch i and data value (Z_i, Y_{i-1}), a control u_i is called optimal if and only if there exists a $U \in \xi$ such that U is optimal and $u_i = U(Z_i,\ Y_{i-1})$.

Also, it is convenient to define the sequence of *optimal value functions* V_i, $i = 0, \ldots, N$, such that

$$V_i(Z_i, Y_{i-1}) = \inf_{U \in \xi} E_{/Z_i, Y_{i-1}}\left[\frac{1}{2}\, s_N x_N^2 + \sum_{j=i}^{N-1} \frac{1}{2}(a_j x_j^2 + b_j u_j^2)\right] \tag{125}$$

when (Z_i, Y_{i-1}) is such that this expectation exists, where u_j denotes $U(Z_j, Y_{j-1})$. Heuristically, V_i is the usual conditional expected cost-to-go function at epoch i, given the currently available data but before the control u_i is used.

5.2 Equation System II for State Estimation

For the purpose of analyzing this control problem, it is more convenient to adopt a somewhat different system of equations for generating approximate moments of the conditional state distribution, even though this requires some effort to establish some relationships between the two equation systems. This other system, denoted Equation System II, is the following for $i = 0, \ldots, N-1$:

$$d_{i+1} = \left(\frac{r_{i+1}}{\mu_{i+1} + r_{i+1}}\right)^2 \nu_{i+1} + \frac{\lambda_{i+1}}{r_{i+1}} e_{i+1}\,; \; d_0 = 0; \tag{126}$$

$$\hat{x}_{i+1} = f_i \hat{x}_i + u_i + \left(\mu_{i+1} + \frac{2r_{i+1}\nu_{i+1}}{\mu_{i+1} + r_{i+1}}\right)\left(\frac{e_{i+1}}{\mu_{i+1} + r_{i+1}}\right) + \frac{\lambda_{i+1}}{r_{i+1}^2}\left(e_{i+1}^2 - \mu_{i+1} - r_{i+1}\right);$$

$$\hat{x}_0 = \bar{x}_0; \tag{127}$$

and

$$\lambda_{i+1} = \left(\frac{r_{i+1}}{\mu_{i+1} + r_{i+1}}\right)^3 (f_i^3\lambda_i + 3f_i p_i q_i \psi_i); \; \lambda_0 = 0, \tag{128}$$

where μ_{i+1} and p_i are as defined by Eqs. (3) and (4), and where

$$\nu_{i+1} = f_i^2 d_i + q_i \psi_i \hat{x}_i \tag{129}$$

and

$$e_{i+1} = z_{i+1} - f_i \hat{x}_i - u_i. \tag{130}$$

In this system, the μ_i, p_i, and λ_i are actually just predetermined parameter sequences; only the d_i and $\hat{x}_i$ are random variables a priori. This system is constructed to give the following first-order approximations to the variables generated by Equation System I, in the form of Eqs. (7), (8), (10), (11), (16), and (17):

$$\hat{x}_i \approx \bar{x}_i,$$

$$p_i + 2d_i \approx v_i,$$

$$\lambda_i \approx \bar{\lambda}_i,$$

and

$$\mu_i + 2\nu_i \approx m_i.$$

In the present context, of course, the u_i values are not generally a known sequence a priori, because the control law is allowed to generate them by feedback from the state measurements. Because of the nonanticipative nature of this feedback, however, all *previous* control values are known at each epoch, so the state estimation results of Section 4 can clearly be applied to this case as well if we take $N = i$ at each epoch i.

5.3 Formally First-Order Optimal Control Law

The analysis of Ref. 1 can be applied here to give approximations to an optimal control law and the value functions which are formally accurate to first order in h. In this particular case the approximation for the control law reduces to

$$\hat{u}_i = -\frac{s_{i+1} f_i \hat{x}_i + \phi_{i+1}}{s_{i+1} + b_i}; \quad i = 0, \ldots, N-1; \tag{131}$$

where $\hat{u}_i$ denotes the control generated by this law at epoch i, $\hat{x}_i(Z_i, Y_i)$ as generated by Eqs. (126) to (130), and the other parameters are defined by the recursions

$$s_k = a_k + \frac{f_k^2 b_k s_{k+1}}{b_k + s_{k+1}}; \quad s_N \text{ as given}; \tag{132}$$

$$y_k = f_k^2 \left[\frac{s_{k+1}^2}{b_k + s_{k+1}} + \left(1 - \frac{p_{k+1}}{r_{k+1}}\right)^2 y_{k+1} \right]; \quad y_N = 0; \tag{133}$$

and

$$\phi_k = \frac{f_k b_k \phi_{k+1}}{b_k + s_{k+1}} + \left[s_{k+1} + \left(1 - \frac{p_{k+1}}{r_{k+1}}\right)^2 y_{k+1} \right] q_k \psi_k; \quad \phi_N = 0; \tag{134}$$

with $k = N-1, N-2, \ldots, 0$ and with p_{k+1} as defined earlier by Eqs. (3) and (4).

The corresponding approximation to the optimal value functions is given by

$$J_i(Z_i, Y_{i-1}) = \frac{1}{2} s_i(\hat{x}_i^2 + p_i + 2d_i) + \phi_i \hat{x}_i + y_i d_i + \frac{1}{2}\eta_i, \tag{135}$$

where η_i now denotes the value defined by the recursion (for $k = N-1, \ldots, 0$)

$$\eta_k = \eta_{k+1} + s_{k+1} q_k + \frac{f_k^2 p_k s_{k+1}^2}{b_k + s_{k+1}}; \quad \eta_N = 0. \tag{136}$$

To allow for later modifications, this is taken as the definition of J_i only for $(Z_i, Y_{i-1}) \in G_i$, in which case the expectations in Eq. (125) for V_i certainly exist. Hence, we can also define the error functions ϵ_i for such (Z_i, Y_{i-1}) as

$$\epsilon_i(Z_i, Y_{i-1}) = V_i(Z_i, Y_{i-1}) - J_i(Z_i, Y_{i-1}), \quad i = 0, \ldots, N. \tag{137}$$

The objective in this section is to show that, except possibly for the unlikely set of realizations mentioned earlier, the control $\overline{u}_i(Z_i, Y_{i-1})$ differs from the optimal control by less than $h^{5/4}$, or more precisely, that if there is a control law U_1 such that

$$|U_1(Z_i, Y_{i-1}) - \hat{u}_i(Z_i, Y_{i-1})| > h^{5/4},$$

then there exists a control law $U_2 \in U$ such that

$$|U_2(Z_i, Y_{i-1}) - \hat{u}_i(Z_i, Y_{i-1})| < h^{5/4}$$

and

$$J(U_2) \leqslant J(U_1).$$

The derivation of the result in this latter form does not require the actual existence or uniqueness of an optimal control law.

5.4 Relations between Equation Systems I and II

Equation System II can be constructed if we regard d_i as one half of $\nu - p_i$, then expand Equation System I (in the form using $\overline{\lambda}_i$ and λ_i^* in place of $\overline{\eta}_i$ and η_i^*) formally to first order in h, with the quantities ψ_i, $\overline{\lambda}_i$, λ_i^*, and d_i being of order h, and finally relabel $\tilde{x}_i$ and $\underline{\lambda}_i$ as $\hat{x}_i$ and λ_i to distinguish them from their exact values. If the realization through an epoch i is such that $(Z_i, Y_{i-1}) \in G_i$, therefore, it

is a straightforward but tedious matter to use Taylor's theorem with remainder, Conditions (1) to (4) of Section 3.1 for $(Z_i, Y_{i-1}) \in G_i$, and the fact that $h^{-1/16}$ can be made arbitrarily large compared to $\ln(1/h)$ if we take h sufficiently small, to establish the following result by induction on the index j of these four conditions in Section 3.1.

Lemma 11: There exists an $h^* > 0$ depending only on the parameters a, b, and F of the unperturbed problem, such that if $h \leqslant h^*$ and $(Z_i, Y_{i-1}) \in G_i$ for some epoch i, then

$$|\hat{x}_j| < h^{-1/16},$$
$$|d_j| < h^{15/16},$$
$$|\hat{x}_j - \bar{x}_j| \leqslant F_j h^2,$$
$$|p_j + 2d_j - v_j| \leqslant F_j h^2,$$

and

$$|\lambda_j - \bar{\lambda}_j| \leqslant F_j h^2$$

for $j = 0, \ldots, i-1$, where F_j grows slowly enough with j that $F_j < h^{-1/4}$ if

$$j \leqslant i \leqslant \frac{1}{32}\sqrt{\ln(1/h)}.$$

The last conclusion here also uses the inequalities noted at the beginning of Section 4.5. Furthermore, applying the inequalities of Lemma 1 and the results in Appendix C enables us to obtain the following additional inequalities during the course of the same induction.

Lemma 12: There exists an $h^* > 0$ whose value depends only on the parameters a, b, and F, such that if $h \leqslant h^*$ and if (Z_i, Y_{i-1}) and $(Z_i, Y_{i-1})' \in G_i$ differ in at most one component, whose values are denoted here as ρ and ρ', then

$$|[\hat{x}_j(\rho) - \bar{x}_j(\rho)] - [\hat{x}_j(\rho') - \bar{x}_j(\rho')]| \leqslant F_j h^2 |\rho - \rho'|$$

and

$$|[v_j(\rho) - 2d_j(\rho)] - [v_j(\rho') - 2d_j(\rho')]| \leqslant F_j h^2 |\rho - \rho'|$$

for $j = 0, \ldots, i-1$, where F_j is the same as in Lemma 11.

Again, the computational details of the proof are too lengthy and routine to be given here. As before, the notation $\hat{x}_i(\rho')$ is an abbreviation for $\hat{x}_j[(Z_j, Y_{j-1})']$, etc., where $(Z_j, Y_{j-1})'$ is the obvious initial segment at epoch j of $(Z_i, Y_{i-1})'$. These results can now be combined with those of Lemmas 1 and 10 to conclude the following results for Equation System II:

Lemma 13: There exists an $h^* > 0$ whose value depends only on the parameters a, b, and F of the unperturbed estimation problem, such that if $h \leqslant h^*$ and if (Z_i, y_{i-1}) and $(Z_i, Y_{i-1})' \in G_i$ differ in at most one component, whose values are denoted here as ρ and ρ' respectively, and if $i < \frac{1}{32}\sqrt{\ln(1/h)}$, then

$$|e_i| < \sqrt{b + Q}\ \sqrt{9\ln(1/h)} < h^{-1/8},$$

$$|\hat{x}_i - \bar{x}_i| < \left\{1 + 5\sqrt{\frac{b}{\pi}}\right\} h^{7/4},$$

$$|\sigma_i^2 - p_i - 2d_i| < (1 + 5b\sqrt{2})\, h^{7/4},$$

$$|\lambda_i - \theta_i| < \left(1 + 9b\sqrt{\frac{b}{\pi}}\right) h^{7/4},$$

and

$$|\gamma_i - 3p_i^2| < b^2 h + 6b(h^{7/4} + h^{7/8}) < h^{3/4}$$

for either realization, and

$$|[\bar{x}_i(\rho) - \hat{x}_i(\rho)] - [\bar{x}_i(\rho') - \hat{x}_i(\rho')]| < \sqrt{b}\left(6 + 2i\sqrt{\frac{2}{\pi}}\, C_i\right) h^3$$

$$+ \left(1 + F_i + 2i\sqrt{\frac{2b}{\pi}}\, D_i\right) h^2 |\rho - \rho'|,$$

$$|[\sigma_i^2(\rho) - 2d_i(\rho)] - [\sigma_i^2(\rho') - 2d_i(\rho')]| < 2b(1 + \sqrt{2}\, C_i) h^3$$

$$+ (1 + F_i + 2b\sqrt{2}\, D_i)\, h^2 |\rho - \rho'|,$$

$$|d_i(\rho) - d_i(\rho')| < h^{3/4} |\rho - \rho'|,$$

and

$$|\hat{x}_i(\rho) - \hat{x}_i(\rho')| < |\rho - \rho'|.$$

Proof: To establish the first inequality, $f_{i-1}\tilde{x}_{i-1}$ is added and subtracted from the definition of e_i to give (for $i \geqslant 1$)

$$e_i = (z_i - f_{i-1}\tilde{x}_{i-1} - u_{i-1}) - f_{i-1}(\hat{x}_{i-1} - \tilde{x}_{i-1}).$$

Thus, from the triangle inequality and Eq. (7),

$$|e_i| \leqslant |z_i - x_i^*| + |f_{i-1}||\hat{x}_{i-1} - \tilde{x}_{i-1}|.$$

Since $(Z_i,\ Y_{i-1})$ or $(Z_i,\ Y_{i-1})' \in G_i$,

$$|z_i - x_i^*| < \sqrt{m_i + r_i}\, \sqrt{8 \ln(1/h)} < \sqrt{b + Q}\, \sqrt{8\, \ln(1/h)}$$

by definition and $|\hat{x}_{i-1} - \tilde{x}_{i-1}| < h^{7/8}$ by Lemma 11 for sufficiently small positive h. So, since $|f_i| < 1$ and $h^{7/8} < \sqrt{(b + Q) \ln(1/h)}$ for sufficiently small positive h,

$$|e_i| < \sqrt{b + Q}\, \sqrt{9 \ln(1/h)}$$

for sufficiently small positive h.

The next four inequalities can clearly be verified from the triangle inequality and the results of Lemmas 10 and 11 by similar constructions. Likewise, the next two (Lipschitz) inequalities of this lemma follow from Lemmas 10 and 12; and the next to the last inequality from Lemmas 1 and 12, since $\Gamma_i < h^{-1/4}$ by the recursion for Γ_i and the remarks at the beginning of Section 4.5.

For the last inequality, it follows from Eq. (127) that, for $k = 0, \ldots, i-1$,

$$\hat{x}_{k+1} - \hat{x}'_{k+1} = f(\hat{x}_k - \hat{x}_k') + \frac{\mu}{\mu + r}(e - e') + \frac{2r}{(\mu + r)^2}(\nu e - \nu' e') + \frac{\lambda}{r^2}(e^2 - e'^2),$$

where $\hat{x}_k'$denotes $\hat{x}_k[(Z_k,\ y_{k-1})']$, etc., and where the obvious subscripts of the other quantities are omitted. From the definition of e_{k+1},

$$\hat{x}_{k+1} - \hat{x}_k = f\left(\frac{r}{\mu + r}\right)(\hat{x}_k - \hat{x}_k') + \frac{\mu}{\mu + r}(z_{k+1} - z_{k+1}')$$

$$+ \frac{2r}{(\mu + r)^2} [\nu(e - e') + e'(\nu - \nu')] + \frac{\lambda}{r^2}(e + e')(e - e').$$

Repeated application of the triangle inequality and the inequalities already established show that the last two terms in this expression have magnitudes less than $\sqrt{h}\ |\rho - \rho'|$ for sufficiently small positive h if $|\hat{x}_k - \hat{x}_k'| < |\rho - \rho'|$, since (Z_k, Y_{k+1}) and $(Z_k, Y_{k-1}) \in G_k$. Hence, again by the triangle inequality, if $|\hat{x}_k - \hat{x}_k'| < |\rho - \rho'|$, then

$$|\hat{x}_{k+1} - \hat{x}'_{k+1}| < \left[\left| f \left(\frac{r}{\mu + r} \right) \right| + 2\sqrt{h} \right] |\rho - \rho'| \text{ if } \rho \text{ does not denote } z_{k+1}$$

and

$$|\hat{x}_{k+1} - \hat{x}_{k+1}| \left(\frac{\mu}{\mu + r} + 2\sqrt{h} \right) |\rho - \rho'| \text{ if } \rho \text{ denotes } z_{k+1}.$$

Since $|f| < 1$, $\mu > 0$, and $r > 0$,

$$|\hat{x}_{k+1} - \hat{x}'_{k+1}| < |\rho - \rho'|$$

in either case for all $h \leqslant \bar{h}$, where $\bar{h}$ is some strictly positive number not depending on the index k. Since the desired inequality is trivially true for $k = 0$, it holds for all $k \leqslant i - 1$ by induction. □

5.5 Some Loose Bounds

Before proceeding further, it is helpful to establish some relatively loose bounds on the value function and, when it exists, the optimal control at a generic epoch i.

Lower Bound on Value Function

If the problem is altered so that the controller is given exact knowledge of the current state as well as the noisy measurements thereof at and after epoch $i+1$ and is allowed to use this extra data in the control law in a Borel-measurable way, this can only expand the class of admissible control laws. Hence, minimizing over this expanded class cannot raise the conditional expected cost-to-go at epoch i.

For epoch $k \geqslant i + 1$, define $H_k\ (Z_k, Y_{k-1}, x_{i+1}, \ldots, x_k)$ as the optimal expected cost-to-go at epoch k in the altered problem, i.e.,

$$H_k\ (Z_k, Y_{k-1}, x_{i+1}, \ldots, x_k) = \inf_{\bar{U} \in \bar{\xi}} E \left[\frac{1}{2} s_n x_n^2 + \sum_{j=k}^{N-1} \frac{1}{2}(a_j x_j^2 + b_j u_j^2) \right],$$

where u_j is generated from $Z_j, Y_{j-1}, x_{i+1}, \ldots, x_j$ by the control law $\bar{U}$, the expectation is conditioned on the same data, and $\bar{\xi} \supseteq \xi$ is the new class of admissible control laws just described. From the form of the dynamics of Eq. (1), H_k depends only on x_k. Now suppose that

$$H_{k+1} = \frac{1}{2} g x_{k+1}^2 + \alpha x_{k+1} + \frac{1}{2}\beta.$$

Then, by the standard principle of optimality of dynamic programming,

$$H_{k+1} = \inf_u E_{/x} \left\{ \frac{1}{2}(ax^2 + bu^2) + \frac{1}{2} g [fx + u + (1 + \psi_x) w]^2 \right.$$
$$\left. + \alpha [fx + u + (1 + \psi x) w] + \frac{1}{2} \beta \right\},$$

where the k subscripts have been suppressed. Evaluating the expectations gives

$$H_k = \inf_u \left\{ \frac{1}{2}(ax^2 + bu^2) + \frac{1}{2} g[(fx+u)^2 + q(1+\psi x)^2] + \alpha(fx+u) + \frac{1}{2}\beta \right\}.$$

The infimum can be attained by the use of

$$u = -\frac{gfx+\alpha}{b+g},$$

which gives (with additional k subscripts deleted)

$$H_k(x) = \frac{1}{2}[ax^2 + gf^2x^2 + g\,q\,(1+\psi x)^2 - \frac{(g\,fx+\alpha)^2}{b+g} + \beta] + \alpha\,fx$$

$$= \frac{1}{2}\left[\left(\frac{f^2b}{b+g} + q\,\psi^2\right)g + a\right]x^2 + \left(\frac{fb\alpha}{b+g} + gq\,\psi\right)x + \frac{1}{2}\left(\beta + gq - \frac{\alpha^2}{b+g}\right).$$

Since $H_N(x_N) = \frac{1}{2} s_N x_N^2$ for $i = N-1$, it follows by reverse induction on i that

$$H_{i+1}(x) = \frac{1}{2}\, g_{i+1}\, x^2 + \alpha_{i+1}\, x + \frac{1}{2}\beta_{i+1},$$

where g_{i+1}, α_{i+1}, and β_{i+1} are determined by the reverse recursions

$$g_k = \frac{f_k^2 b_k\, g_{k+1}}{b_k + g_{k+1}} + g_{k+1} q_k \psi_k^2 + a_k; \quad g_N = s_N; \tag{138}$$

$$\alpha_k = \frac{f_k b_k \alpha_{k+1}}{b_k + g_{k+1}} + g_{k+1}\, q_k\, \phi_k; \quad \alpha_N = 0; \tag{139}$$

and

$$\beta_k = \beta_{k+1} + g_{k+1}\, q_k - \frac{\alpha_{k+1}^2}{b_k + g_{k+1}}; \quad \beta_N = 0. \tag{140}$$

For a general u_i followed thereafter by use of an optimal control law, a lower bound on the conditional expected cost-to-go at epoch i is therefore

$$\frac{1}{2}[\,b_i\, u_i^2 + a_i(\bar{x}_i^2 + \sigma_i^2)] + E[H_{i+1}(x_{i+1})\,/Z_i,\ U_{i-1} * u_i].$$

Suppressing i subscripts, this expectation is

$$E\left\{\frac{1}{2}\, g_{i+1}\, [fx + u + (1+\psi x)w]^2 + \alpha_{i+1}[fx + u + (1+\psi x)w] + \frac{1}{2}\,\beta_{i+1}\,/Z_i, U_i\right\}.$$

Evaluating this expectation as the composition of a marginal and a conditional given x, we get

$$\frac{1}{2}\, g_{i+1}\, [(f\bar{x} + u)^2 + q\,(1+\psi\bar{x})^2 + (f^2 + q\psi^2)\sigma^2] + \gamma_{i+1}(f\bar{x}+u) + \frac{1}{2}\beta_{i+1}.$$

This conditional expected cost-to-go can be rewritten as

$$\frac{1}{2}(b_i + g_{i+1})(u_i - \bar{u}_i)^2 + \frac{1}{2}\, g_i(\bar{x}_i^2 + \sigma_i^2) + \alpha_i\,\bar{x}_i + \frac{1}{2}\left(\beta_i + \frac{f_i^2 g_{i+1}\sigma_i^2}{b_i + g_{i+1}}\right),$$

where

$$\bar{u}_i = -\frac{f_i g_{i+1}\,\bar{x}_i + \alpha_{i+1}}{b_i + g_{i+1}}, \tag{141}$$

and α_i, β_i, g_i, and g_{i+1} are given by Eqs. (138) to (140).

By construction, this is a lower bound on the conditional expected cost-to-go at epoch i in the original problem for any control u_i, even if it is followed by use of an optimal control law at epochs $i+1, \ldots, N$. Also, by construction and the definition of the value function V_i, u_i can be chosen as $\bar{u}_i$ (a function of Z_i and Y_{i-1}, actually) to give

$$V_i(Z_i, Y_{i-1}) \geqslant \frac{1}{2} g_i(\bar{x}_i^2 + \sigma_i^2) + \alpha_i \bar{x}_i + \frac{1}{2}\left[\beta_i + \frac{f_i^2\, g_{i+1}\sigma_i^2}{b_i + g_{i+1}}\right] \tag{142}$$

for all $(Z_i, Y_{i-1}) \in G_i$, where $\bar{x}_i$ and σ_i^2 are the conditional mean and variance of x_i defined earlier, which have been shown to exist for realizations such that $(Z_i, Y_{i-1}) \in G_i$, where Y_{i-1} is now the sequence $\{\bar{u}_0(Z_0), \bar{u}_1(Z_1, Y_0), \ldots, \bar{u}_{i-1}, (Z_{i-1}, Y_{i-2})\}$.

Upper Bound on Value Function

Returning to the original control problem, we now consider the obviously admissible control law which gives

$$u_k = \begin{cases} -\hat{u}_k \text{ (as defined in Eq. 131)};\ k < i \\ \bar{u}_k \text{ if } k = i \\ -\dfrac{g_{k+1} f_k z_k + \alpha_{k+1}}{b_k + g_{k+1}};\ k \geqslant i+1 \end{cases} \tag{143}$$

and, for $k \geqslant i+1$, denote the corresponding conditional expected cost-to-go by

$$T_k(Z_k, Y_{k-1}) = E\left[\frac{1}{2} s_n x_n^2 + \sum_{j=k}^{n-1} \frac{1}{2}(a_j x_j^2 + b_j u_j^2) / Z_k, Y_{k-1}\right]$$

when (Z_k, Y_{k-1}) is such that this expectation exists. For $k \geqslant i+1$,

$$u_k = -c_k(x_k + n_k) - \delta_k,$$

with

$$c_k = \frac{g_{k+1} f_k}{b_k + g_{k+1}}$$

and

$$\delta_k = \frac{\alpha_{k+1}}{b_k + g_{k+1}},$$

so, dropping obvious subscripts,

$$x_{k+1} = (f - c)\, x_k - \delta - cn + (1 + \psi x)\, w.$$

For $k \geqslant i+1$ and $(Z_i, Y_{i-1}) \in G_i$, let $\bar{x}_k$ temporarily denote $E(x_k / Z_i, Y_{i-1})$ under the control law of Eq. (143). By the statistical independence of the process and measurement noise variables,

$$E(x_{k+1} / Z_i, x_k) = (f - c)\, x_k - \delta,$$

so

$$E(x_{k+1} / Z_i) = (f - c)\, \bar{x}_k - \delta = \bar{x}_{k+1}, \tag{144}$$

where the dependence on Y_{k-1} is suppressed in the notation. Also,

$$E(x_{k+1}^2 / x_k, Z_k) = (f - c)^2 x_k^2 - 2\delta(f - c) x_k + \delta^2 + c^2 r_k + (1 + \psi x_k)^2 q_k,$$

so from Eq. (144) and noise independence

$$\sigma_{k+1}^2 = [(f-c)^2 + \psi^2 q]\,\sigma_k^2 + c^2 r(1+\psi\bar{x}_k)^2 q, \tag{145}$$

where σ_k^2 now denotes the conditional variance of x_k given Z_k.

By definition,

$$T_k(Z_i, Y_{i-1}) = T_{k+1}(Z_i, Y_{i-1}) + \frac{1}{2} E_{/Z_i, Y_{i-1}} \{a_k x_k^2 + b_k [c_k(x_k + n_k) + \delta_k]^2\}$$

for $k \geq i+1$, or with notational abbreviations,

$$T_k = T_{k+1} + \frac{1}{2}\{a_k(\bar{x}_k^2 + \sigma_k^2) + b_k[c_k^2(\bar{x}_k^2 + \sigma_k^2 + r_k) + 2c_k\delta_k\bar{x}_k + \delta_k^2]\}. \tag{146}$$

Now suppose that

$$T_{k+1} = \frac{1}{2} g_{k+1}(\bar{x}_{k+1}^2 + \sigma_{k+1}^2) + \alpha_{k+1}\bar{x}_{k+1} + \frac{1}{2}(\beta_{k+1} + \xi_{k+1}), \tag{147}$$

where g_{k+1}, α_{k+1}, and β_{k+1} are given by Eqs. (138) to (140) and ξ_{k+1} is given by the recursion

$$\xi_j = \xi_{j+1} + \frac{f_j^2 g_{j+1} r_j}{b_j + g_{j+1}};\ \xi_N = 0. \tag{148}$$

Since this supposition is true at $k = N-1$, T_k is given by Eq. (147) for all $k \geq i+1$, as can be shown by backwards induction on k with the use of Eq. (146).

The conditional expected cost-to-go at epoch i using an arbitrary u_i, but the control law of Eq. (143) thereafter, is therefore

$$E_{/Z_i, Y_{i-1}}\left[\frac{1}{2}(a_i x_i^2 + b_i u_i^2) + \frac{1}{2} g_{i+1}(\bar{x}_{i+1}^2 + \sigma_{i+1}^2) + \alpha_{i+1}\bar{x}_{i+1} + \frac{1}{2}(\beta_{i+1} + \xi_{i+1})\right]$$

$$= \frac{1}{2}[a_i(\bar{x}_i^2 + \sigma_i^2) + b_i u_i^2] + \frac{1}{2} g_{i+1}(\bar{x}_{i+1}^2 + \sigma_{i+1}^2) + \alpha_{i+1}\bar{x}_{i+1} + \frac{1}{2}(\beta_{i+1} + \xi_{i+1}) \tag{149}$$

by idempotence of expectation, at least for $(Z_i, Y_{i-1}) \in G_i$ and sufficiently small positive h, when these moments are known to exist. Also,

$$x_{i+1} = f_i x_i + u_i + (1 + \psi_i x_i)\, w_i,$$

so

$$\bar{x}_{i+1} = f_i \bar{x}_i + u_i$$

and

$$\sigma_{i+1}^2 = (f_i^2 + q_i\psi_i^2)\sigma_i^2 + (1 + \psi_i\bar{x}_i)^2 q_i,$$

where $\bar{x}_i$ and σ_i^2 now have their former meaning. By definition, this conditional expected cost-to-go is an upper bound on $V_i(Z_i, Y_{i-1})$ for any u_i. Using Eq. (141) therefore gives, for sufficiently small positive h,

$$V_i(Z_i, Y_{i-1}) \leq \frac{1}{2}(b_i + g_{i+1})\,(u_i - \bar{u}_i) + \frac{1}{2} g_i(\bar{x}_i^2 + \sigma_i^2) + \alpha_i\bar{x}_i$$

$$+ \frac{1}{2}\left[\beta_i + \xi_{i+1} + \frac{f_i^2 g_{i+1}^2 \sigma_i^2}{b_i + g_{i+1}}\right] \tag{150}$$

for all $(Z_i, Y_{i-1}) \in G_i$, where $\bar{x}_i$ and σ_i^2 are functions of (Z_i, Y_{i-1}), and $i \leq N-1$.

Limits on Optimal Control

With $u_i = \bar{u}_i$ in the upper bound (150), the principle of optimality and the lower bound on V_i preceding Eq. (141) imply that $u_i = u$ can be bettered (in the sense of achieving a lower conditional expected cost-to-go) unless

$$(b_i + g_{i+1})\,(u - \bar{u}_i)^2 \leqslant \xi_{i+1},$$

at least for $(Z_i, Y_{i-1}) \in G_i$ and for sufficiently small positive h. Since

$$\xi_{i+1} = \sum_{k=i+1}^{N-1} \frac{f_k^2 g_{k+1}^2 r_k}{b_k + g_{k+1}},$$

this condition implies that $(u - \hat{u}_i)^2 < \frac{1}{2}\ln(1/h)$ for sufficiently small positive h and $N < \frac{1}{32}\sqrt{\ln(1/h)}$ if the recursion (138) for g_k is backward stable. This is clearly true for small enough positive h, since

$$g_k = \left[f_k^2\left(\frac{b_k}{b_k + g_{k+1}}\right) + q_k\psi_k^2\right] g_{k+1} + a_k.$$

Also, comparing these parameters with the $J\,(Z_i, U_{i-1})$ parameters gives

$$(g_k - s_k) = \left(\frac{f_k b_k}{b_k + g_{k+1}}\right)\left(\frac{f_k b_k}{b_k + s_{k+1}}\right)(g_{k+1} - s_{k+1}) + g_{k+1}\, q_k\, \psi_k^2;\quad (g_N - s_N) = 0$$

(so $g_k \geqslant s_k > 0$) and

$$\begin{aligned}(\gamma_k - \phi_k) = {} & \left(\frac{f_k b_k}{b_k + g_{k+1}}\right)(\gamma_{k+1} - \phi_{k+1}) \\ & + \left[q_k\,\psi_k - \frac{f_k b_k \phi_{k+1}}{(b_k + g_{k+1})\,(b_k + s_{k+1})}\right](g_{k+1} - s_{k+1}) \\ & - \left(\frac{r_{k+1}}{m_{k+1} + r_{k+1}}\right)^2 q_k\,\psi_k\; Y_{k+1}\,;\quad (\gamma_N - \phi_N) = 0.\end{aligned}$$

Hence, for sufficiently small positive h, $|g_{i+1} - s_{n+1}| < h^{3/2}$ and $|\gamma_{i+1} - \phi_{i-1}| < \sqrt{h}$. Using $\hat{u}_i$ to denote the apparent approximately optimal control as before, i.e.,

$$\hat{u}_i = -\frac{f_i\, s_{i+1}\,\hat{x}_i + \phi_{i+1}}{b_i + s_{i+1}},$$

it follows easily that, for small enough $h > 0$,

$$|\hat{u}_i - \bar{u}_i| < h^{-1/2}|\,\hat{x}_i - \bar{x}_i| + h^{3/2}\,\hat{x}_i + \sqrt{h}.$$

From Lemma 10, therefore,

$$|\hat{u}_i - \bar{u}_i| < 2h + \sqrt{h} < 2\sqrt{h} \text{ if } |\hat{x}_i| < \frac{1}{\sqrt{h}} \text{ and } (Z_i, Y_{i-1}) \in G_i,$$

or equivalently for sufficiently small positive h, by the definition of G_i. This means that under these conditions u_i cannot be optimal (i.e., can be bettered by some other value) unless

$$|u_i - \hat{u}_i| < \sqrt{\ln(1/h)}.$$

The preceding results can be summarized as follows.

Lemma 14: For any $c > 0$, there exists an $h^* > 0$ such that if $h \leqslant h^*$, if the number N of epochs in the control problem of Section 5.1 is such that

$$N < \frac{1}{32}\sqrt{\ln(1/h)},$$

and if $(Z_i, Y_{i-1}) \in G_i$ for some $i \in \{0, 1, \ldots, N\}$, then

$$0 \leqslant V(Z_i, Y_{i-1}) - \frac{1}{2} g_i (\bar{x}_i^2 + \sigma_i^2) + \alpha_i \bar{x}_i + \frac{1}{2}\left|\beta_i + \frac{f_i^2 g_{i+1} \sigma_i^2}{b_i + g_{i+1}}\right| \leqslant \xi_{i+1},$$

where $\bar{x}_i (Z_i, Y_{i-1})$ and $\sigma_i^2 (Z_i, Y_{i-1})$ are the conditional mean and variance of x_i given Z_i, and where g_i, α_i, β_i, and ξ_{i+1} are defined by Eqs. (138) to (140) and (148). Also, for any $U \in \xi$, if

$$|U(Z_i, Y_{i-1}) - \hat{u}_i (Z_i, Y_{i-1})| \geqslant c\sqrt{\ln(1/h)},$$

then there exists a $u \in \mathbb{R}$ such that $W_i (Z_i, Y_{i-1}, u) < W_i [Z_i, Y_{i-1}, U(Z_i, Y_{i-1})]$, where the expected cost-to-go W_i is defined as

$$W_i (Z_i, Y_{i-1}, u) = \inf_{U \in \xi} E_{/Z_i, Y_{i-1}} \left[\frac{1}{2}(s_n x_n^2 + a_i x_i^2 + b_i u^2) + \sum_{k=i+1}^{N-1} \frac{1}{2} (a_k x_k^2 + b_k u_k^2)\right],$$

with $u_k = U(Z_k, Y_{k-1})$ for $k = i + 1, \ldots, N - 1$.

Bound On $|\epsilon_{i+1}|$

At this point it is convenient to extend the definition of J_{i+1} for all arguments (Z_{i+1}, Y_i) for which the initial segment $(Z_i, Y_{i-1}) \in G_i$. If $(Z_{i+1}, Y_i) \notin G_{i+1}$, J_{i+1} is defined as

$$J_{i+1} (Z_{i+1}, Y_i) = E\left[\frac{1}{2} s_N^2 x_N^2 + \sum_{k=i+1}^{N-1} \frac{1}{2} (a_k x_k^2 + b_k u_k^2)/Z_{i+1}, Y_i\right], \tag{151}$$

where

$$u_k = -\frac{f_k g_{k+1} z_k + \alpha_{k+1}}{b_k + g_{k+1}} \text{ for } k \geqslant i + 1. \tag{152}$$

Using the result of Eq. (105) in the derivation of Eq. (149) shows that this extension of J_{i+1} is well defined. In this case, J_{i+1} is defined as the conditional expected cost-to-go using an admissible but possibly nonoptimal control law, namely the one used to establish Eq. (149). Hence, V_{i+1} is well defined by Eq. (125) for such $(Z_{i+1}, Y_i) \notin G_i$, and the definition of ϵ_{i+1} can be extended to this case as

$$\epsilon_{i+1} (Z_{i+1}, Y_i) = V_{i+1} (Z_{i+1}, Y_i) - J(Z_{i+1}, Y_i). \tag{153}$$

Since Eq. (149) is valid in this case as well,

$$0 \leqslant J_{i+1} (Z_{i+1}, Y_i) - V_{i+1} (Z_{i+1}, Y_{i-1}) = -\epsilon_{i+1} (Z_{i+1}, Y_i) \leqslant \xi_{i+1} \tag{154}$$

for sufficiently small positive h and any $i = 0, \ldots, N - 1$. If $N \leqslant \frac{1}{32}\sqrt{\ln(1/h)}$, $(Z_i, Y_{i-1}) \in G_i$ and $(Z_{i+1}, Y_i) \notin G_{i+1}$. In this case, therefore,

$$|\epsilon_{i+1} (Z_{i+1}, Y_i)| < \frac{1}{2} \ln(1/h) \tag{155}$$

for sufficiently small positive h.

5.6 Principle of Optimality for Generic Epoch i

Applying the standard principle of optimality of dynamic programming to the optimal value function of Eq. (125), we get

$$V_i(Z_i,\ Y_{i-1}) = \inf_u E\left\{\frac{1}{2}(a_i x_i^2 + b_i u^2) + V_{i+1}[Z_i * z_{i+1}(u),\ Y_{i-1} * u]\right\},$$

where the expectation is over x_i, w_i, and n_{i+1} and is conditioned on Z_i and $Y_{i-1} * u$, and where $z_{i+1}(u)$ denotes the quantity $f_i x_i + u + (1 + \psi_i x_i)\ w_i + n_{i+1}$.

Furthermore, if $(Z_i,\ Y_{i-1}) \in G_i$ the expectation in this expression exists and Eq. (153) can be used to give

$$V_i(Z_i,\ Y_{i-1}) = \inf_u \underset{x_i, w_i, n_{i+1}/Z_i, Y_{i-1}, u}{E} \Big\{ J_{i+1}[Z_i * z_{i+1}(u),\ Y_{i-1} * u]$$

$$+ \epsilon_{i+1}[Z_i * z_{i+1}(u),\ Y_{i-1} * u] + \frac{1}{2}(a_i x_i^2 + b_i u^2)\Big\}. \qquad (156)$$

Also, the infimum in Eq. (156) can be restricted to the set

$$A_i(Z_i,\ Y_{i-1}) = \{u\colon |u - \hat{u}_i(Z_i, Y_{i-1})| < \sqrt{\ln(1/h)}\},$$

in this case by virtue of Lemma 14, which, since $(Z_i,\ Y_{i-1}) \in G_i$, is contained in the set

$$\{u\colon |u| < 2\ \ln(1/h)\}$$

for sufficiently small positive h, by Eq. (131).

If $\bar{J}_i(Z_i,\ Y_{i-1})$ is defined as in Eq. (135) for *all* $(Z_i,\ Y_{i-1})$, then it coincides with J_i if $(Z_i,\ Y_{i-1}) \in G_i$. For $(Z_i,\ Y_{i-1}) \in G_i$ but $(Z_{i+1},\ Y_i) \notin G_{i+1}$, however,

$$J_{i+1} - \bar{J}_{i+1} = \frac{1}{2}[g_{i+1}(\bar{x}_{i+1}^2 + \sigma_{i+1}^2) - s_{i+1}(\hat{x}_{i+1}^2 + p_{i+1} + 2d_{i+1})] + \alpha_{i+1}\,\bar{x}_{i+1}$$

$$- \phi_{i+1}\,\hat{x}_{i+1} - y_{i+1}\,d_{i+1} + \frac{1}{2}\left[\beta_{i+1} + \xi_{i+2} + \frac{f_{i+1}\,g_{i+2}^2\,\sigma_{i+1}^2}{b_{i+1} + g_{i+2}} - \eta_{i+1}\right]. \qquad (157)$$

It is straightforward to verify that $(Z_i,\ Y_{i-1}) \in G_i$ and $|u_i - \hat{u}_i(Z_i,\ Y_{i-1})| < \sqrt{\ln(1/h)}$ imply that

$$(Z_{i+1},\ Y_i) \in G_{i+1} \Leftrightarrow (Z_{i+1} - x_{i+1}^*)^2 < 8(m_{i+1} + r_{i+1})\ \ln(1/h).$$

Therefore, Eq. (156) can be replaced, for $(Z_i,\ Y_{i-1}) \in G_i$ and for all sufficiently small positive h, by

$$V(Z_i,\ Y_{i-1}) = \inf_{u \in A_i(Z_i, Y_{i-1})} \Big\{E[\bar{J}_{i+1}(Z_i * z_{i+1},\ Y_{i-1} * u)$$

$$+ \epsilon_{i+1}(Z_i * z_{i+1},\ Y_{i-1} * u) + \frac{1}{2}(a_i x_i^2 + b_i u^2)]$$

$$+ \int_I [J_{i+1}(Z_i * \rho,\ Y_{i-1} * u) - \bar{J}_{i+1}(Z_i * \rho,\ Y_{i-1} * u)]\ dP_{z_{i+1}}(\rho)\Big\}, \qquad (158)$$

where the first expectation and the probability measure $P_{z_{i+1}}$ are conditional on Z_i, Y_{i-1}, and u as in Eq. (156), and

$$I = \{\rho\colon (\rho - x_{i+1}^*)^2 > 8(m_{i+1} + r_{i+1})\ln(1/h)\}.$$

It is a fairly straightforward but tedious matter to conclude that the integral term in Eq. (158) is smaller in magnitude than h^3 for any $u \in A_i(Z_i, Y_{i-1})$ and sufficiently small positive h by the use of Eq. (157), previously established error bounds, and inequality (105) for large values of $|\rho - x^*_{i+1}|$. As a result, it follows from further straightforward manipulation that

$$V_i(Z_i, Y_{i-1}) = P_1(Z_i, Y_{i-1}) + P_2(i) + P_3(Z_i, Y_{i-1}) + P_4(Z_i, Y_{i-1}) + \inf_{u \in A_i(Z_i, Y_{i-1})} H_0(Z_i, Y_{i-1}, u) \tag{159}$$

$$\text{for } (Z_i, Y_{i-1}) \in G_i,\ h \text{ sufficiently small}, i \leqslant N - 1 < \frac{1}{32}\sqrt{\ln(1/h)},$$

where

$$\begin{aligned} P_1 = {} & \frac{1}{2}\hat{x}_i^2(a_i + f_i^2 s_{i+1}) + \frac{1}{2}\left[\eta_{i+1} + s_{i+1}p_{i+1} + (f_i^2 p_i + q_i + r_{i+1})\left(\frac{p_{i+1}}{r_{i+1}}\right)^2 s_{i+1}\right] \\ & + \hat{x}_i\left[f_i\phi_{i+1} + 2\left(\frac{p_{i+1}q_i\psi_i}{\mu_{i+1} + r_{i+1}}\right)s_{i+1} + \left(\frac{p_{i+1}}{r_{i+1}}\right)^2 q_i\psi_i s_{i+1} \right. \\ & \left. + (y_{i+1} + s_{i+1})\left(\frac{r_{i+1}}{\mu_{i+1} + r_{i+1}}\right)^2 q_i\psi_i\right] \\ & + f_i^2 d_i\left[(y_{i+1} + s_{i+1})\left(\frac{r_{i+1}}{\mu_{i+1} + r_{i+1}}\right)^2 + \frac{2s_{i+1}p_{i+1}}{\mu_{i+1} + r_{i+1}} + \left(\frac{p_{i+1}}{r_{i+1}}\right)^2\right], \end{aligned} \tag{160}$$

and where P_2, P_3, and P_4 are rather complicated expressions, for which it follows from the inequalities of Lemmas 11 to 13, the results of Appendix C, and the backwards stability of Eq. (132) that P_2 is independent of (Z_i, Y_{i-1}),

$$|P_2| < h^{7/4},\ |P_3| < h^{5/2}, \tag{161}$$

and, if $(Z_i, Y_{i-1})' \in G_i$ differs from (Z_i, Y_{i-1}) in only one component, whose values in (Z_i, Y_{i-1}) and $(Z_i, Y_{i-1})'$ are denoted by ρ and ρ' respectively,

$$|P_4(\rho) - P_4(\rho')| \leqslant h^{-7/16}(h^3 + h^2|\rho - \rho'|), \tag{162}$$

and where

$$\begin{aligned} H_0(Z_i, Y_{i-1}, u) = {} & \frac{1}{2}(b_i + s_{i+1})u^2 + [s_{i+1}f_i\hat{x}_i + \phi_{i+1} + C(Z_i, Y_{i-1})]u \\ & + \delta_i(Z_i, Y_{i-1}, u), \end{aligned} \tag{163}$$

with

$$\begin{aligned} C(Z_i, Y_{i-1}) = \frac{s_{i+1}}{r_{i+1}}\Bigg(& \left[p_{i+1} + 2\left(\frac{r_{i+1}}{\mu_{i+1} + r_{i+1}}\right)^2 \nu_i\right](\bar{x}_i - \hat{x}_i) \\ & + \frac{\lambda_{i+1}}{r_{i+1}}\{f_i^2[\sigma_i^2 - p_i - (\bar{x}_i - \hat{x}_i)^2] + q_i\psi_i[2\bar{x}_i + \psi_i(\bar{x}_i^2 + \sigma_i^2)]\}\Bigg) \end{aligned} \tag{164}$$

and

$$\delta_i(Z_i, Y_{i-1}, u) = E_{/Z_i, Y_{i-1}, u}[\epsilon_{i+1}(Z_i * z_{i+1}(u), Y_{i-1} * u)]. \tag{165}$$

Applying the inequalities of Lemma 13 and the results in Appendix C on composite Lipschitz conditions to Eq. (164) we can show, after some routine manipulation, that for sufficiently small positive h,

$$|C(Z_i, Y_{i-1})| \leqslant h^{3/2} \tag{166}$$

and

$$|C[(Z_i, Y_{i-1})] - C[(Z_i, Y_{i-1})']| < h^{-7/16}(h^3 + h^2|\rho - \rho'|) \tag{167}$$

if (Z_i, Y_{i-1}) and $(Z_i, Y_{i-1})'$ are both in G_i and differ in only one component, whose values are denoted ρ and ρ' respectively.

From this point on it is assumed for simplicity that there exists a Borel-measurable $u_0(Z_i, Y_{i-1})$ for all (Z_i, Y_{i-1}), $i = 0, \ldots, N-1$, such that

$$u_i^*(Z_i, Y_{i-1}) = \inf_u H_0(Z_i, Y_{i-1}, u).$$

If such a minimizing u^* does not always exist, the ensuing results can still be obtained by a standard limiting procedure. With this assumption and the definition

$$\tilde{u}_i(Z_i, Y_{i-1}) = -\frac{s_{i+1} f_i \hat{x}_i(Z_i, Y_{i-1}) + \phi_{i+1} + C(Z_i, Y_{i-1})}{b_i + s_{i+1}}, \tag{168}$$

it follows with considerable computation from Eqs. (135), (137), (159), and (163) that

$$\begin{aligned}\epsilon_i(Z_i, Y_{i-1}) = {} & P_2(i) + P_3(Z_i, Y_{i-1}) + P_4(Z_i, Y_{i-1}) + \frac{1}{2}(s_{i+1} + b_i)(u_i^* - \tilde{u}_i)^2 \\ & - \frac{s_{i+1} f_i \hat{x}_i(Z_i, Y_{i-1}) C(Z_i, Y_{i-1})}{s_{i+1} + b_i} - \frac{C(Z_i, Y_{i-1})\left[\phi_{i+1} + \frac{1}{2} C(Z_i, Y_{i-1})\right]}{s_{i+1} + b_i} \\ & + \delta_i\left[Z_i, Y_{i-1}, u_i^*(Z_i, Y_{i-1})\right]\end{aligned} \tag{169}$$

for all $(Z_i, Y_{i-1}) \in G_i$.

It is also convenient to establish the following results for future use.

Lemma 15: There exist an $h^* > 0$ and a $\xi > 0$ (independent of h and epoch index i) such that if

$$h \leqslant h^*,$$

$$i \leqslant N - 1 < \frac{1}{32}\sqrt{\ln(1/h)},$$

$$(Z_i, Y_{i-1}) \in G_i,$$

$$\left|\hat{x}_i(Z_i, Y_{i-1})\right| \leqslant \xi K,$$

$$\left|d_i(Z_i, Y_{i-1})\right| \leqslant \xi K h,$$

$$\left|u_i - \hat{u}_i(Z_i, Y_{i-1})\right| \leqslant \sqrt{\ln(1/h)},$$

$$\left[z_{i+1} - x_{i+1}^*(Z_i, Y_{i-1} * u_i)\right]^2 \leqslant 8(m_{i+1} + r_{i+1})\ln(1/h),$$

and

$$h^{-1/16} \leqslant K \leqslant h^{-1/8},$$

then

$$\left| \hat{x}_{i+1}(Z_i \bullet z_{i+1}, Y_{i-1} \bullet u_i \right| \leqslant K$$

and

$$\left| d_{i+1}(Z_i \bullet z_{i+1}, Y_{i-1} \bullet u_i \right| \leqslant Kh.$$

Proof: This is a routine matter of substituting inequalities already established for the various variables and the definition of G_i into Equation Systems I and II and using the triangle inequality. The desired result then follows from the fact that $h^{-\epsilon} > \ln(1/h)$ for sufficiently small h if $\epsilon > 0$. □

Lemma 16: There exists an $h^* > 0$ such that if $h \leqslant h^*$, $i \leqslant N - 1 < \frac{1}{32}\sqrt{\ln(1/h)}$, and (Z_i, Y_{i-1}) and $(Z_i, Y_{i-1})' \in G_i$ differ in only one component, whose respective values are ρ and ρ', then

$$\left| \tilde{u}_i(\rho) - \tilde{u}_i(\rho') \right| < h^{5/2} + |\rho - \rho'|.$$

Proof: Since $|f_i|$ and $\left| \dfrac{s_{i+1}}{b_i + s_{i+1}} \right|$ are both strictly less than unity, this lemma follows if we apply Lemma 13 and Eq. (167) to Eq. (168) and use the triangle inequality. □

Lemma 17: There exists an $h^* > 0$ such that if $h \leqslant h^*$, $i \leqslant N - 1 < \frac{1}{32}\sqrt{\ln(1/h)}$, and $(Z_i, Y_{i-1}) \in G_i$, then

$$\left| \tilde{u}_i(Z_i, Y_{i-1}) - \hat{u}_i(Z_i, Y_{i-1}) \right| < \frac{1}{2} h^{5/4}.$$

Proof: From Eqs. (131), (168), and (166),

$$\left| \tilde{u}_i - \hat{u}_i \right| = \frac{|C|}{b_i + s_{i+1}} \leqslant \frac{h^{3/2}}{b_i + s_{i+1}},$$

in this case for sufficiently small positive h. Since $b_i \geqslant B > 0$ and $s_{i+1} \geqslant 0$, their sum is less than $h^{-1/4}$ for $h < B^{-4}$. □

Definition: For $h, K > 0$, $i = 0, \ldots, N$, $\pi_i(K)$ is the subset of $(Z_i, Y_{i-1}) \in G_i(h)$ such that

$$|\hat{x}_j(Z_j, Y_{j-1})| \leqslant K$$

and

$$|d_j(Z_j, Y_{j-1})| \leqslant Kh$$

for $j = 0, \ldots, i$, where (Z_j, Y_{j-1}) denotes the indicated initial segment of (Z_i, Y_{i-1}).

The control law $\hat{U}$ is now considered as an approximation to an optimal control law, where

$$\hat{U}(Z_i, Y_{i-1}) = \begin{cases} \hat{u}_i \text{ if } (Z_i, Y_{i-1}) \in \pi_i(h^{-1/16}) \\ -\dfrac{f_i g_{i-1} z_i + \alpha_{i+1}}{b_i + g_{i+1}} \quad \text{otherwise.} \end{cases}$$

It is shown next in Theorem 2 that (Z_i, Y_{i-1}) remains in $\pi_i(h^{-1/16})$ with high prior probability for small h, it will be shown in Section 5.7 that $\hat{u}_i$ is a first-order approximation to an optimal control in this case.

Theorem 2: If the total number of epochs N always satisfies the inequality

$$N < \sqrt{\ln \ln(1/h)}$$

and the control law $\hat{U}$ is used, then, a priori,

$$\lim_{h \to 0^+} \Pr\{(Z_i,\ Y_{i-1}) \in \pi_i(h^{-1/16});\ i = 0,\ \ldots,\ N(h)\} = 1,$$

assuming of course that the parameters a, b, F, s_N, A, and B of the unperturbed problem do not vary with N.

Proof: For the problem at hand, let c and $\bar{h}$ be the positive constants cited in the conditions of Section 3.1, and assume that $c < 2$ without loss of generality. Let $\{\theta_i : i = 0,\ \ldots,\ N-1\}$ be the sequence defined by

$$\theta_{i+1} = \theta_i^{\frac{2}{c}};\ \theta_0 = 1/h \geqslant 1$$

and let ω_i denote $\ln \theta_i$. Now suppose that

$$\max\{|x_0 - \bar{x}_0|,\ |w_0|,\ |n_N|,\ \max_{i=1,\ldots,N-1} \{|w_i|,\ |n_i|\}\} < c\sqrt{\omega_0} \tag{170}$$

and $\max\{|u_j| : j = 0,\ \ldots,\ i-1\} < c\omega_{i-1}$ for some i, with $h < \bar{h}$. By construction, $1 < \theta_0 < \theta_1 < \ldots$, so $\omega_0 < \ln(1/h)$, since the logarithm is monotonic. Therefore, by the conditions of Section 3.1 on the unperturbed problem,

$$(Z_i,\ Y_{i-1}) \in G_i\left(\frac{1}{\theta_{i-1}}\right)$$

if h is also chosen as less than $\bar{h}$. In light of the previously established inequalities, a strictly positive h_1 exists such that if $h \leqslant h_1$, then

$$|\hat{u}_k| < 2\ln(1/h)$$

whenever $(Z_k,\ Y_{k-1}) \in G_k(h)$ and $k < \frac{1}{32}\sqrt{\ln(1/h)}$. Hence if $h = \frac{1}{\theta_0}$ is further chosen to be less then h_1, then

$$|u_i(Z_i,\ Y_{i-1})| < 2\ln\theta_{i-1} < c\ \omega_i.$$

Since $\omega_i > \omega_{i-1}$ by construction, it follows from an obvious induction on i that

$$(Z_n,\ Y_{n-1}) \in G_N\left(\frac{1}{\theta_N}\right)$$

whenever inequality (170) holds. From its definition $h_2 < h_1 \Rightarrow G_N(h_2) \supseteq G_N(h_1)$. So, since $\theta_N > \theta_0 = 1/h$ by construction,

$$(Z_N,\ Y_{N-1}) \in G_N(h)$$

for all h less than some strictly positive value h^*, if $N < \frac{1}{32}\sqrt{\ln(1/h)}$ and

$$\max\{|x_0 - \bar{x}_0|,\ |w_0|,\ |n_N|,\ \max_{i=1,\ \ldots,\ N-1}\{|w_i|,\ |n_i|\}\} < c\sqrt{\ln(1/h)}.$$

Also, $h^* > 0$ can be chosen so that Lemma 11 holds under these conditions, too, in which case

$$(Z_N,\ Y_{n-1}) \in \pi_N(h^{-1/16}).$$

From the bound in Section 3.1, the prior probability P of this event *not* occurring is bounded above by

$$\frac{\sigma^2}{c\sqrt{2\pi}} \left(\frac{N}{\sqrt{\ln \gamma}}\right) \rho^{\frac{N}{2}} e^{-\frac{1}{2}\left(\frac{c}{\sigma}\right)^2 \rho^{-N} \ln\gamma}$$

where $\sigma^2 = \max\{v_0, q_0, r_N, \max_{i=1,\ldots,N-1} \{q_i, r_i\}\}$ and γ and ρ denote $1/h$ and $\frac{2}{c}$, for brevity. Since $N < \sqrt{\ln \ln(1/h)}$, another bound is

$$P < \frac{\sigma^2}{c\sqrt{2\pi}} e^{\frac{1}{2}\left[N \ln\rho - \left(\frac{c}{\sigma}\right)^2 \rho^{-N} \ln\gamma\right]}.$$

To establish that P approaches zero as h does, as long as $N < \sqrt{\ln \ln(1/h)}$, it therefore suffices to show that the limit of the exponential factor

$$N(\gamma) \ln \rho - \left(\frac{c}{\sigma}\right)^2 \rho^{-N(\gamma)} \ln \gamma$$

is zero as $\gamma \to \infty$. Clearly $N(\gamma)$ can be taken as equal to $\sqrt{\ln \ln \gamma}$ for this purpose. The logarithm of the subtrahend in this quantity is

$$2 \ln\left(\frac{c}{\sigma}\right) + \ln \ln \gamma - \ln \rho \sqrt{\ln \ln \gamma},$$

which can be made arbitrarily large for any c and σ by making γ large enough. Also, the ratio of the minuend to the subtrahend is

$$\left(\frac{\sigma}{c}\right)^2 \ln\rho \left(\frac{N}{\rho^{-N} \ln\gamma}\right).$$

The logarithm of the last factor is

$$\ln N + N \ln\rho - \ln\gamma < \ln(\sqrt{\ln\ln\gamma}) + \ln\rho\sqrt{\ln\ln\gamma} - \ln\gamma,$$

which goes to $-\infty$ as $\gamma \to \infty$. Since the other factors are constant, the entire ratio goes to zero as $\gamma \to \infty$. Therefore,

$$\lim_{h\to 0} P(h) = 0.$$

Since $(Z_N, Y_{N-1}) \in S_N(h^{-1/16}) \Rightarrow (Z_i, Y_{i-1}) \in \pi_i(h^{-1/16})$ for all $i \in \{0, \ldots, N\}$, by construction, the theorem follows. □

5.7 Induction Argument

In this section, an induction argument is developed which demonstrates as a corollary that the control $\hat{u}_i$ generated by control law $\hat{U}$ is optimal to order h for realizations in π_i $(h^{-1/16})$ for sufficiently small h. For a generic epoch $i \leq N-1$, the induction hypothesis is that, for some $h^* > 0$ which does not depend on the epoch index i, if

$h \leq h^*$,

$N < \frac{1}{32}\sqrt{\ln(1/h)}$,

$Q_{i+1} \in [h^{-1/16}, h^{-1/8}]$,

and (Z_{i+1}, Y_i), $(Z_{i+1}\ Y_i)' \in \pi_{i+1}$ (Q_{i+1}) differ in only one component, whose values are denoted as ρ and ρ' respectively,

then

$$|\epsilon_{i+1}(\rho) - \epsilon_{i+1}(\rho')| < K_{i+1} h^{-1/2} (h^3 + h^2 |\rho - \rho'|); \quad h^{-1/16} \leq K_{i+1} \leq h^{-1/8}.$$

Here, $\epsilon_{i+1}(\rho)$ is shorthand for $\epsilon_{i+1}[(Z_{i+1}, Y_i)]$, etc.

Lemma 18: If the preceding induction hypothesis holds, then there exists an $\tilde{h} > 0$ (independent of h^*) such that if $h \leq \tilde{h}$, $Q_i = \xi Q_{i+1}$, ξ as in Lemma 15, (Z_i, Y_{i-1}) and $(Z_i, Y_{i-1})' \in \pi_i(Q_i)$ and differ in only one component, whose respective values are ρ and ρ' respectively, and if

$$|u_i - \hat{u}_i(\rho)|, \ |u_i' - \hat{u}_i(\rho')| \leq \sqrt{\ln(1/h)},$$

then

$$|\delta_i(\rho, u_i) - \delta_i(\rho', u_i')| \leq \left(\frac{1}{8} + 4K_{i+1} h^{-1/2} + \ln^2(1/h)\, C_{i+1}\right) h^3$$

$$+ K_{i+1} h^{3/2} |u - u'| + (2K_{i+1} h^{-1/2} + \ln^2(1/h) D_{i+1}) h^2 |\rho - \rho'|.$$

Proof: Let

$$t_1 = \frac{z_{i+1} - \bar{z}_1}{\sqrt{m_{i+1}(\rho) + r_{i+1}}} \ ; \ \bar{z}_1 = f_i \bar{x}_i(\rho) + u_i$$

and

$$t_2 = \frac{z_{i+1} - \bar{z}_2}{\sqrt{m_{i+1}(\rho') + r_{i+1}}} \ ; \ \bar{z}_2 = f_i \bar{x}_i(\rho') + u_i'.$$

By definition, then

$$\delta_i(\rho, u_i) - \delta_i(\rho', u_i') = \underset{t_1/Z_i(\rho'), Y_{i-1}(\rho'), u_i'}{E} \{\epsilon_{i+1}[Z_i(\rho') \bullet (\bar{z}_1 + t_1\sqrt{m_{i+1}(\rho) + r_{i+1}}), \ Y_{i-1}(\rho) \bullet u_i]\}$$

$$- \underset{t_2/Z_i(\rho'), Y_{i-1}(\rho'), u_i'}{E} \{\epsilon_{i+1}[Z_i(\rho') \bullet (\bar{z}_2 + t_2\sqrt{m_{i+1}(\rho') + r_{i+1}}), Y_{i-1}(\rho') \bullet u_i']\}.$$

By construction, $t_1 = s(\rho)$ and $t_2 = s(\rho')$ in the context of Section 4.4, so by Eq. (96) and integration of Eq. (95) over $|s| < \epsilon M_i - 6\sqrt{\ln(1/h)}$,

$$\Pr\{|t_j| > \sqrt{8 \ln(1/h)}\} < \frac{2h^4}{\sqrt{\ln(1/h)}} \ ; \ j = 1, 2;$$

for sufficiently small $h > 0$ (because $M_i > \frac{1}{4} h^{-1/8}$), this probability being conditioned on $(Z_i, Y_{i-1}) \in G_i(h)$ and on u_i. This result does not depend on any restrictions on u_i. For sufficiently small positive h, therefore, given such (Z_i, Y_{i-1}),

$$\Pr\{|t_1| \sqrt{m_{i+1}(\rho) + r_{i+1}} > 8\sqrt{\ln(1/h)}\} < \frac{h^3}{4 \ln(1/h)},$$

and

$$\Pr\{|t_2| \sqrt{m_{i+1}(\rho') + r_{i+1}} > 8\sqrt{\ln(1/h)}\} < \frac{h^3}{4 \ln(1/h)}.$$

By inequality (155), $|\epsilon_{i+1}| < \frac{1}{2}\ln(1/h)$ in such cases (for sufficiently small positive h). Since the conditional density of t_1 and t_2 exist otherwise, it follows from the triangle inequality and the use of t to denote both variables of integration that

$$|\delta(\rho, u) - \delta(\rho', u')| < \frac{1}{8}h^3 + \int_{-\sqrt{8\ln(1/h)}}^{\sqrt{8\ln(1/h)}} |\epsilon_{i+1}[Z * (\bar{z}_1 + t\sqrt{m+r}), Y * u] p_{t_1/\rho,u}(t)$$

$$-\epsilon_{i+1}[Z' * (\bar{z}_2 + t\sqrt{m'+r}), Y' * u']|\, p_{t_2/\rho',u'}\, d\lambda_t, \tag{171}$$

where some obvious epoch subscripts are suppressed in the notation. By construction, $p_{t_j/\rho',u}(t)$, $j = 1, 2$, are independent of the value of u, so the integrand of Eq. (171) can be rewritten as the absolute value of

$$\epsilon_{i+1}[Z * (\bar{z}_1 + t\sqrt{m+r}), Y * u][p(t/\rho) - p(t/\rho')]$$
$$+ p(t/\rho')\{\epsilon_{i+1}[Z * (\bar{z}_1 + t\sqrt{m+r}), Y * u] - \epsilon_{i+1}[Z' * (\bar{z}_2 + t\sqrt{m'+r}), Y' * u']\}. \tag{172}$$

The first term in this expression is bounded in magnitude by

$$\frac{1}{2}\ln(1/h)(C_{i+1}h^3 + D_{i+1}h^2|\rho - \rho'|)\left[\frac{e^{-\frac{t^2}{2} + 2|t|}}{\sqrt{2\pi}}\right]$$

from inequality (155), Lemma 1, and the proof of Theorem 1. By construction,

$$|t_1| < \sqrt{8\ln(1/h)} \Leftrightarrow |z_{i+1} - x^*_{i+1}(\rho)| < \sqrt{8[m_{i+1}(\rho) + r_{i+1}]\ln(1/h)}$$

and

$$|t_2| < \sqrt{8\ln(1/h)} \Leftrightarrow |z_{i+1} - x^*_{i+1}(\rho')| < \sqrt{8[m_{i+1}(\rho') + r_{i+1}]\ln(1/h)}.$$

By hypothesis and Lemma 15, therefore,

$$[Z_i(\rho) * z_{i+1}, Y_{i-1}(\rho) * u_i] \in \pi_i(Q_i)$$

and

$$[Z_i(\rho') * z_{i+1}, Y_{i-1}(\rho') * u_i'] \in \pi_i(Q_i)$$

for such z_{i+1}. Hence, by the induction hypothesis and the triangle inequality, the other term in (172) is bounded in magnitude by

$$p(t/\rho')K_{i+1}h^{-1/2}[4h^3 + h^2(|\rho - \rho'| + |\bar{z}_1 - \bar{z}_2| + |t||\sqrt{m} - \sqrt{m'}| + |u - u'|)]$$

for sufficiently small positive h. Also,

$$|\bar{z}_1 - \bar{z}_2| = |f_i[\tilde{x}_i(\rho) - \tilde{x}_i(\rho')] + u_i - u_i'| \leqslant |\tilde{x}_i(\rho) - \tilde{x}_i(\rho')| + |u_i - u_i'|$$

and, by the proof of Lemma 1,

$$|\sqrt{m(\rho)} - \sqrt{m(\rho')}| = \left|\frac{m(\rho) - m(\rho')}{\sqrt{m(\rho)} + \sqrt{m(\rho')}}\right| \leqslant \frac{\Gamma_{i+1}}{2\sqrt{b}} h|\rho - \rho'|,$$

where $\Gamma_{i+1} < h^{-1/4}$ if $i < \frac{1}{32}\sqrt{\ln(1/h)}$. Combining these results and substituting the bound for (172) in the integral of inequality (171) we obtain the conclusion of this lemma for sufficiently small positive h after some routine computation. □

Lemma 19: If the preceding induction hypothesis holds and $(Z_i,\ Y_{i-1}) \in \pi_i(Q_i)$, with Q_i as in Lemma 18, then there exists an $\tilde{h} > 0$ such that if $h \leqslant \tilde{h}$

$$|u_i^*(Z_i,\ Y_{i-1}) - \tilde{u}_i(Z_i,\ Y_{i-1})| < 4\sqrt{\frac{K_{i+1}}{b_i + s_{i+1}}})\ h^{3/2}$$

for all $(Z_i,\ Y_{i-1}) \in \pi_i(Q_i)$.

Proof: Since $(Z_i,\ Y_{i-1}) \in \pi_i(\xi\ Q_{i+1}) \Rightarrow (Z_i,\ Y_{i-1}) \in G_i$, $|\tilde{u}_i - \hat{u}_i| < \sqrt{\ln(1/h)}$ and $|u_i^* - \hat{u}_i| < \sqrt{\ln(1/h)}$ by Lemma 14 if h^* is chosen to be less than the value needed in that lemma. So it follows from Lemma 18 that

$$|\delta_i[u_i^*(Z_i,\ Y_{i-1})] - \delta_i[\tilde{u}_i(Z_i,\ Y_{i-1})]| < 5K_{i+1}h^{-1/2}(h^3 + h^2|u_i^* - \tilde{u}_i|) \tag{173}$$

for all sufficiently small positive h.

Now assume that $|u_i^* - \tilde{u}_i| > \left(\sqrt{\dfrac{10\ k_{i+1}}{s_{i+1} + b_i}} + \dfrac{2}{b_i}\right) h^{3/2}$. Then, dropping epoch subscripts,

$$\begin{aligned}
\frac{1}{2}(s + b)\ (u^* - \tilde{u})^2 &> \frac{1}{2}(s + b)\left(\sqrt{\frac{10\,K}{s + b}} + \frac{2}{b}\right) h^{3/2}\ |u^* - \tilde{u}| \\
&> \frac{1}{2}(s + b)\ h^{3/2}\left(\sqrt{\frac{10\,K}{s + b}}\ |u^* - \tilde{u}| + \frac{2}{b}\ |u^* - \tilde{u}|\right) \\
&> \frac{1}{2}(s + b)\ h^{3/2}\left(\sqrt{\frac{10\,K}{s + b}}\sqrt{\frac{10\,K}{s + b}}\ h^{3/2} + \frac{2}{b}\ |u^* - \tilde{u}|\right) \\
&> \frac{1}{2}(s + b)\ h^3\left(\frac{10\,K}{s + b}\right) + \frac{1}{2}\left(\frac{s + b}{b}\right) 2h^{3/2}\ |u^* - \tilde{u}| \\
&> 5Kh^{-1/2}(h^3 + h^2\ |u^* - \tilde{u}|).
\end{aligned}$$

By inequality (173) and the triangle inequality, therefore, under this assumption

$$\delta_i(\tilde{u}_i) - \delta_i(u_i^*) < \frac{1}{2}(s_{i+1} + b_i)\ (u_i^* - \tilde{u}_i)^2. \tag{174}$$

Let

$$H_1(Z_i,\ Y_{i-1},\ u) = H_0(Z_i,\ Y_{i-1},\ u) - \delta_i(Z_i,\ Y_{i-1},\ u). \tag{175}$$

For the values of Z_i and Y_{i-1} under consideration here,

$$\tilde{u}_i = \arg\min_u \{H_1(u)\}$$

by construction. Also, for a general u,

$$H_1(u) = -\frac{1}{2}\ \frac{(s_{i+1}f_i\hat{x}_i + \phi_{i+1} + C)^2}{s_{i+1} + b_i} + \frac{1}{2}(s_{i+1} + b_i)\ (u - \tilde{u}_i)^2. \tag{176}$$

Therefore, deleting epoch subscripts,

$$\begin{aligned}
\frac{1}{2}(s + b)\ (u^* - \tilde{u})^2 > \delta(\tilde{u}) - \delta(u^*) &\Rightarrow H_1(u^*) + \frac{1}{2}\ \frac{(sf\hat{x} + \phi + C)^2}{s + b} + \delta(u^*) > \delta(\tilde{u}) \\
&\Rightarrow H_1(u^*) + \delta(u^*) > -\frac{1}{2}\ \frac{(sf\hat{x} + \phi + C)^2}{s + b} + \delta(\tilde{u})
\end{aligned}$$

$$\Rightarrow H_0(u^*) > H_0(\tilde{u}) \text{ by Eqs. (175) and (176)).}$$

Combining this result with inequality (174) and Eq. (159) shows that

$$|u_i^* - \tilde{u}_i| \leqslant \left[\sqrt{\frac{10\,K_{i+1}}{s_{i+1} + b_i}} + \frac{2}{b_i}\right] h^{3/2},$$

because the reverse inequality implies that u_i^* is nonoptimal (strictly less optimal than $\tilde{u}_i$, in fact). Since $K_{i+1} \geqslant h^{-1/16}$ by assumption, the lemma follows for all h less than some sufficiently small positive value. □

Lemma 20: If the preceding induction hypothesis holds and $(Z_i, Y_{i-1}) \in \pi_i(Q_i)$, with Q_i as in Lemma 18, then $E\tilde{h} > 0$, such that if $h \leqslant \tilde{h}$,

$$|u_i^*[(Z_i, Y_{i-1})] - u_i^*[(Z_i, Y_{i-1})']| < h + |\rho - \rho'|.$$

Proof: By the triangle inequality,

$$|u_i^*(\rho) - u_i^*(\rho')| \leqslant |u_i^*(\rho) - \tilde{u}_i(\rho)| + |\tilde{u}_i(\rho) - \tilde{u}_i(\rho')| + |\tilde{u}_i(\rho') - u_i^*(\rho')|.$$

The lemma then follows by Lemmas 16 and 19. □

Lemma 21: If the preceding induction hypothesis holds and (Z_i, Y_{i-1}) and $(Z_i, Y_{i-1})' \in \pi_i(Q_i)$, with Q_i as in Lemma 18, differ in only one component, whose respective values are ρ and ρ' respectively, then

$$|\delta_i[\rho,\, u_i^*(\rho)] - \delta_i[\rho' - u_i^*(\rho')]| < \left[\frac{1}{8} + 5K_{i+1}h^{-1/2} + \ln^2\left(\frac{1}{h}\right) C_{i+1}\right] h^3$$

$$+ [3K_{i+1}h^{-1/2} + \ln^2(1/h)\, D_{i+1}] h^2\, |\rho - \rho'|.$$

Proof: From Lemma 14, $|u_i^*(\rho) - \hat{u}_i(\rho)| \leqslant \sqrt{\ln(1/h)}$ and

$$|u_i^*(\rho') - \hat{u}_i(\rho')| \leqslant \sqrt{\ln(1/h)}$$

for sufficiently small positive h. Substituting $u_i^*(\rho)$ for u and $u_i^*(\rho')$ for u' in the inequality of Lemma 18, and using Lemma 20 and the triangle inequality, we can establish the lemma. □

Theorem 3: There exists an $h^* > 0$ which depends only on the parameters a, b, F, S_N, A, and B of the unperturbed problem, such that if

$$h \leqslant h^*;$$

$$N < \frac{1}{32}\sqrt{\ln(1/h)};$$

$$Q_i = \xi Q_{i+1};\; i = 0, \ldots, N-1;$$

and

$$Q_N = h^{-1/8};$$

where ξ is the constant required by Lemma 15, and (Z_i, Y_{i-1}) and $(Z_i, Y_{i-1})' \in \pi_i(Q_i)$ and differ in only one component, whose values are denoted here by ρ and ρ' respectively, for any $i \in \{0, \ldots, N\}$,

then

$$|\epsilon_i(\rho) - \epsilon_i(\rho')| < K_i h^{-1/2}(h^3 + h^2|\rho - \rho'|),$$

where K_i is determined by the recursion

$$K_i = 8K_{i+1}; \quad K_N = h^{-1/16}.$$

Proof (reverse induction on i): By definition, given the premises

$$V_n(Z_N, Y_{N-1}) = \frac{1}{2} s_N E(x_N^2 / Z_N, Y_{N-1})$$
$$= \frac{1}{2} s_N (\bar{x}_N^2 + \sigma_N^2)$$

and

$$J_N(Z_N, Y_{N-1}) = \frac{1}{2} s_N(\hat{x}_N^2 + p_N + 2d_N),$$

since $(Z_N, Y_{N-1}) \in \pi_i(Q_N) \Rightarrow (Z_n, Y_{N-1}) \in G_N$.

Thus,

$$\epsilon_N(Z_N, Y_{N-1}) = \frac{1}{2} s_N [(\bar{x}_N + \hat{x}_N)(\bar{x}_N - \hat{x}_N) - p_N + (\sigma_N^2 - 2d_N)],$$

and the conclusion of the theorem follows directly by application of the triangle inequality and the inequalities of Lemmas 11 and 13, if h^* is sufficiently small.

If the theorem holds at epoch $i+1$, then the initial segments (Z_i, Y_{i-1}) and $(Z_i, Y_{i-1})'$ are both in $G_i(h)$. Since the maximum value of sufficiently small positive h required for inequalities (161), (162), (166), and (167) and those of Lemmas 11, 13, 19, and 21 depend only on the parameters a, b, F, and B of the unperturbed problem, $h^* > 0$ in this theorem can always be chosen as the minimum of these values. Substituting these inequalities into Eq. (169) and using the results in Appendix C for Lipschitz conditions of composite functions, we can show that this theorem holds at epoch i, as long as N is small enough that the recursion generating K_i does not make $K_i > h^{-1/8}$ for $K_N = h^{-1/16}$. But

$$K_i \leqslant K_0 = 8^N h^{-1/16},$$

so

$$\ln K_i \leqslant N \ln 8 + \frac{1}{16} \ln (1/h) < \frac{1}{8} \ln (1/h)$$

for sufficiently small $h^* > 0$ and $h \leqslant h^*$, or

$$K_i < h^{-1/8}$$

in exponential form. □

Corollary: Given the premises of the theorem,

$$|u_i^*(Z_i, Y_{i-1}) - \hat{u}_i(Z_i, Y_{i-1})| < h^{5/4}$$

if h^* is sufficiently small.

Proof: By the triangle inequality,

$$|u_i^* - \hat{u}_i| \leqslant |u_i^* - \bar{u}_i| + |\bar{u}_i - \hat{u}_i|.$$

Since $(Z_i, Y_{i-1}) \in \pi_i(Q_i) \Rightarrow (Z_i, Y_{i-1}) \in G_i$, the corollary follows by Lemmas 17 and 19. □

Even if a strictly optimal control law does not exist, it can still be similarly established via a limiting procedure that if under the premises of Theorem 3

$$|U(Z_i, Y_{i-1}) - \hat{u}_i(Z_i, Y_{i-1})| > h^{5/4}$$

for any admissible control law U, then there exists a $u \in \mathbb{R}$ such that

$$W_i(Z_i, Y_{i-1}, u) < W_i\,[Z_i, Y_{i-1}, U(Z_i, Y_{i-1})],$$

where W is as defined in Lemma 14. Hence the admissible control law U', constructed from U by replacing $U\,(Z_i, Y_{i-1})$ with $\hat{u}_i(Z_i, Y_{i-1})$, is such that

$$J(U') \leqslant J(U).$$

6. CONCLUSIONS

One is often interested in systems which are controllable but unstable in open-loop operation (i.e., with $u_i \equiv 0$). Strictly speaking, this situation is not covered here because $|f_i| < 1$ in Eq. (1). This is not really a limitation, however, because such a problem could always be reformulated in terms of deviations from a stabilizing control law of the form

$$u_i = -c_i z_i\,, \quad 0 < f_i - c_i < 1,$$

in which case the dynamics of Eq. (1) become

$$x_{i+1} = (f_i - c_i)\,x_i + \tilde{u}_i + (1 + \psi_i\,x_i)\,w_i - c_i\,n_i,$$

where

$$\tilde{u}_i = u_i + c_i\,z_i$$

now plays the role of the control variable. The analysis could then proceed as before, but with some extra terms appearing. This extra generality was not included because the analysis was already very complicated and was only intended to be exploratory.

Other interesting extensions of the results here, or similar ones, would be to the multivariable and continuous-time contexts and to steady-state behavior in infinite-time problems. As a first step toward analyzing the continuous-time case, one might consider adapting the approach used here to a discretized problem of the type described in Ref. 1, where there is a time-increment parameter Δ (i.e., the time between successive epochs) which is small compared to the perturbation parameter h. The results obtained would at least have a formal bearing on the limiting contnuous-time problem, and might be suggestive for conducting a mathematically precise analysis of it. This sort of procedure, however, seems to require the use of a third-order Taylor series expansion in carrying out the propagation step of Section 4.3, and with the sort of constructions used here, it did not produce any useful results in the updating step of Section 4.4.

7. REFERENCES

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Appendix A
NONEXISTENCE OF PROBABILITY DENSITY

For simplicity, we consider only the particular case of a single transition where

$$y = x + (1 + hx)\omega,$$

with x and ω independent zero-mean unit-variance normal random variables, and show that the probability density of y diverges at $-1/h$. From the constructions of Section 4.3 and their role in the overall estimation problem, the same reasoning can be applied to more general examples.

If the new random variable w is defined as

$$w = (1 + hx)\omega,$$

then

$$y = x + w$$

and, except at $(x,\omega) = (-1/h, 0)$,

$$p(x,w) = p(x)\, p(w/x) = \frac{e^{-\frac{x^2}{2} - \frac{w^2}{2(1+hx)^2}}}{2\pi\,|1 + hx|}.$$

Hence, for positive $\epsilon < \dfrac{1}{3h}$,

$$\Pr\left\{-\frac{1}{h} - \epsilon \leq y < -\frac{1}{h} + \epsilon\right\} \geq \int_{2\epsilon}^{\frac{1}{h}-\epsilon} \int_{-w-\frac{1}{h}-\epsilon}^{-w-\frac{1}{h}+\epsilon} p(x,w)\, dx\, dw.$$

For each value of w in this region of integration,

$$p(x,w) \geq \left[\frac{1}{2\pi h(w + \epsilon)}\right] e^{-\frac{1}{2}\left[\left(w+\frac{1}{h}+\epsilon\right)^2 + \frac{1}{h^2}\left(\frac{w}{w-\epsilon}\right)^2\right]}.$$

Since $2\epsilon \leq w \leq \dfrac{1}{h} - \epsilon \Rightarrow \left|\dfrac{w}{w - \epsilon}\right| \leq 2$ and $\left|w + \dfrac{1}{h} + \epsilon\right| \leq \dfrac{2}{h^2}$,

$$p(x,w) \geq \left[\frac{e^{-\frac{4}{h^2}}}{2\pi h}\right] \frac{1}{(w + \epsilon)}$$

in this region. Substituting this bound in the preceding integral and changing the variable of integration over w to

$$u = w + \epsilon$$

gives

$$\Pr\left\{y \in \left(-\frac{1}{h} - \epsilon, -\frac{1}{h} + \epsilon\right)\right\} \geq \frac{\epsilon}{\pi h}\, e^{-\frac{4}{h^2}} \int_{3\epsilon}^{1/h} \frac{du}{u}.$$

Thus

$$\frac{1}{2\epsilon}\,\Pr\left\{y \in \left(-\frac{1}{h} - \epsilon,\, -\frac{1}{h} + \epsilon\right)\right\} \geqslant \frac{e^{-4/h^2}}{\pi h}\left[\ln\left(\frac{1}{h}\right) + \ln\left(\frac{1}{3\epsilon}\right)\right]$$

for all positive ϵ less than $1/h$. For any fixed $h > 0$, this lower bound approaches infinity as $\epsilon \to 0$. Hence, the probability density function of y diverges at $-1/h$.

Appendix B
SPECIFIC CASE

In the context of Section 3.1, assume that inequality (13) holds with $k = 12$, $\bar{h} = e^{-5}$, and $c = \frac{1}{50}$; that $\tilde{x}_0 = 0$, and $\nu_0 = 1$; and that $f_i = 1/2$ and $q_i = r_{i+1} = 1$ for $i = 0, \ldots, N-1$. Assume that $h \leqslant \bar{h}$. Since $h \leqslant \bar{h}$, $\frac{d}{dh}[h \ln(1/h)] = \ln(1/h) - 1 > 0$, so

$$h \ln(1/h) \leqslant 5e^{-5} < 0.04.$$

Also, $p_i = \frac{\mu_i}{\mu_i + 1}$ and $\mu_{i+1} = \frac{1}{4}p_i + 1$. Hence, by an easy induction,

$$\left.\begin{array}{l} \mu_{i+1} \text{ is in the interval } (1, 5/4) \\ p_i \in (1/2, 1) \end{array}\right\} i = 0, \ldots, N-1,$$

$$a = \frac{1}{2k} = \frac{1}{24},$$

and

$$b = \frac{5}{4}k = 15.$$

Now assume that Conditions 1 through 4 of Section 3.1 are met at generic epoch i (i.e., that for this realization $(Z_i, Y_{i-1}) \in G_i$) and also that $|x_i - \tilde{x}_i| < \sqrt{\ln(1/h)}$ for this realization. We next verify that this implies the same conditions at epoch $i+1$. This is done in two steps. First, Eqs. (7) to (9) are used to establish bounds for x^*_{i+1}, m_{i+1}, η^*_{i+1}, $|x_{i+1} - x^*_{i+1}|$, and $\frac{(z_{i+1} - x^*_{i+1})^2}{m_{i+1} + r_{i+1}}$, then Eqs. (10) to (12) are used to establish bounds on $\tilde{x}_{i+1}$, ν_{i+1}, η_{i+1}, and $|x_{i+1} - \tilde{x}_{i+1}|$. The obvious epoch subscripts are dropped in the notation.

Step 1:

By the triangle inequality and the assumed bounds,

$$|x^*| = |f\tilde{x} + u| \leqslant \frac{1}{2}|\tilde{x}| + |u| \leqslant 0.52 \ln(1/h) < \ln(1/h).$$

Since $\frac{\nu}{m} \in [0, 1]$ and $(1 + \psi\tilde{x}) \in (0.96, 1.04)$, it likewise follows from Eq. (9) that

$$|\eta^*| < \frac{1}{8}\left(|\eta| + \frac{3h}{\frac{1}{4}\sqrt{\nu}} \times \frac{4}{3}\right)$$

$$< \frac{1}{8}\left(3\sqrt{b} + \frac{16}{\sqrt{a}}\right)h$$

$$< 3\sqrt{b}\,h, \text{ since } a = \frac{1}{24} \text{ and } b = 15.$$

Since $|x_i - \tilde{x}| \leqslant \sqrt{\ln(1/h)}$ by assumption, it follows from Eqs. (1) and (7) that

$$|x_{i+1} - x^*| \leqslant \frac{1}{2}|x_i - \tilde{x}| + |1 + \psi\tilde{x}||w|$$

$$\leqslant \left[\frac{1}{2} + (1.04)(0.02)\right]\sqrt{\ln(1/h)} < 0.521\sqrt{\ln(1/h)},$$

and therefore that

$$|z - x^*| \leqslant |x_i - x^*| + |v| \leqslant 0.541\sqrt{\ln(1/h)}$$

and

$$\frac{(z - x^*)^2}{m + r} = \frac{(z - x^*)}{m+1} < 8\ln(1/h).$$

From Eqs. (3) and (8), it follows that

$$m - \mu = \frac{1}{4}(v - p) + 2\psi\tilde{x} + \psi^2\tilde{x}^2.$$

Since $v - p$ is in the interval $[a - 1, b - \frac{1}{2}] \subset [-1, b]$ and

$$2\psi\tilde{x} + \psi^2\tilde{x}^2 \in [-2\psi\tilde{x}, 2\psi\tilde{x} + \psi^2\tilde{x}^2] \subset \left[-\frac{1}{2}, \frac{3}{4}\right]$$

by the inequalities assumed,

$$m - \mu = \alpha + \beta, \text{ where } \alpha \in \left[-\frac{1}{4}, \frac{b}{4}\right] \text{ and } \beta \in \left[-\frac{1}{2}, \frac{3}{4}\right].$$

Thus

$$(m - \mu) \in \left[-\frac{3}{4}, \frac{b+3}{4}\right].$$

Since $\mu \in \left[1, \frac{5}{4}\right]$, $m \in [1/4, 6] \subset [a, b]$ for $a = \frac{1}{24}$ and $b = 15$.

Step 2:

From Eq. (12), $|\eta| < |\eta^*| < 3\sqrt{b}\,h$.

From Eq. (11),

$$v = \frac{m}{m+1}\left[1 + \eta^* \frac{\sqrt{m}}{(m + 1)^2} (z - x^*)\right]^2.$$

Setting the derivative to zero shows that $\frac{\sqrt{m}}{(m + 1)^2}$ is maximized when $m = 1/3$, so

$$\frac{\sqrt{m}}{(m + 1)^2} \leqslant \left(\frac{3}{4}\right)^2\sqrt{\frac{1}{3}} < 1/3.$$

Also, since $h < 0.1$, and $h \ln(1/h) < 0.04$,

$$\eta^*(z - x^*) < 3\sqrt{bh}\ (0.541)\sqrt{h \ln(1/h)}$$
$$< 3\sqrt{1.5}\,(0.541)\sqrt{0.04} < 1.$$

Hence,

$$\left[1 + \eta^* \frac{\sqrt{m}}{(m+1)^2}(z - x^*)\right]^2 \in [1/3,\ 2].$$

Since $m \in [1/4, 6]$,

$$\frac{m}{m+1} \in \left[\frac{1}{5}, \frac{6}{7}\right]$$

and

$$v \in \left[\frac{1}{15}, 2\right] \subset [a,b].$$

In Eq. (10),

$$\frac{m}{m+1}|z - x^*| < \left(\frac{6}{7}\right)(0.541)\sqrt{\ln(1/h)} < 0.463\sqrt{\ln(1/h)}.$$

Since $h < 0.007$,

$$\frac{1}{3}\ln(1/h) > 1,$$

and

$$h\left[\frac{(z - x^*)^2}{m+1} - 1\right] \in \left[-h,\ \frac{1}{3}h \ln(1/h) - h\right] \subset [-0.007, 0.007].$$

Also, $\dfrac{m^{3/2}}{(m+1)^2}$ is maximized when $m = 3$, giving $\dfrac{3\sqrt{3}}{16}$. Hence,

$$\eta^* \frac{m^{3/2}}{(m+1)^2}\left[\frac{(z - x^*)^2}{m+1} - 1\right] \in [-0.035, 0.035].$$

Therefore, from the triangle inequality and Eq. (10),

$$|\tilde{x} - x^*| < 0.463\sqrt{\ln(1/h)} + 0.035 < \sqrt{\ln(1/h)} \text{ for } h \leqslant e^{-5},$$
$$|\tilde{x}| \leqslant |x^*| + |\tilde{x} - x^*| < 0.983\sqrt{\ln(1/h)} + 0.035 < \sqrt{\ln(1/h)} \text{ for } h \leqslant e^{-5},$$

and

$$|x_{i+1} - \tilde{x}| \leqslant |x_{i+1} - x^*| + |x^* - \tilde{x}| < 0.984\sqrt{\ln(1/h)} + 0.035 < \sqrt{\ln(1/h)} \text{ for } h < e^{-5}.$$

The desired result follows by induction on i, because the induction hypothesis holds by definition for $i = 1$.

Appendix C
LIPSCHITZ CONDITIONS

If

- $|f_i(x) - f_i(y)| \leqslant A_i + B_i|x - y|;\ x$ and $y \in R_i;\ i = 1, \ldots, n$;
- $|g(x_i, \ldots, x_{i1}, \ldots, x_n) - g(x_1, \ldots, x_{i2}, \ldots, x_n)| < C_i + D_i|x_{i1} - x_{i2}|$;

  x_{i1} and $x_{i2} \in f_i(R_i);\ x_k \in f_k(R_k),\ k \neq i$;

  and
- $h(x) \triangleq g[f_1(x), \ldots, f_n(x)];\ x \in \bigcap_{i=1}^{n} R_i$;

then

$$\begin{aligned} h(x) - h(y) = {} & g[f_1(x), \ldots, f_n(x)] - g[f_1(y), f_2(x), \ldots, f_n(x)] \\ & + g[f_1(y), f_2(x), \ldots, f_n(x)] - g[f_1(y), f_2(y), \ldots, f_n(x)] \\ & \cdot \\ & \cdot \\ & \cdot \\ & + g[f_1(y), \ldots, f_{n-1}(y), f_n(x)] - g[f_1(y), \ldots, f_n(y)]. \end{aligned}$$

By the triangle inequality and the Lipschitz conditions for g:

$$|h(x) - h(y)| \leqslant \sum_{i=1}^{h} (C_i + D_i|f_i(x) - f_i(y)|).$$

By the Lipschitz conditions on the f_i:

$$|h(x) - h(y)| \leqslant \sum_{i=1}^{n} C_i + D_i(A_i + B_i|x - y|) = \sum_{i=1}^{n} (C_i + D_i A_i) + (\sum_{i=1}^{n} D_i B_i)|x - y|.$$

Furthermore, by the mean value theorem:

- If f_i' is continuous on $R_i = [a_i, b_i]$, then the above Lipschitz condition on f_i obtains with $A_i = 0$ and $B_i = \max_{x \in R_i} \{|f_i'(x)|\}$.

- If $\frac{\partial g}{\partial x_i}$ is continuous on $f_i\ ([a_i, b_i])$, then the above Lipschitz conditions on g obtain with $C_i = 0$ and $D_i = \max_{x_i \in f_i([a_i, b_i])} \left\{ \left| \frac{\partial g(x_i)}{\partial x_i} \right| \right\}$.

Specific Case (Product):

$$f_1(x)f_2(x) - f_1(y)f_2(y) = f_1(x)[f_2(x) - f_2(y)] + f_2(y)[f_1(x) - f_1(y)],$$

so

$$|f_1(x)f_2(x) - f_1(y)f_2(y)| \leqslant (A_2|f_1(x)| + A_1|f_2(y)|) + (B_2|f_1(x)| + B_1|f_2(y)|)|x - y|.$$

Appendix D
EXISTENCE AND APPROXIMATION OF RADON-NIKODYM DENSITY

Let R denote the ring of all subsets of real numbers which have the form

$$\bigcup_{i=1}^{n} [a_i, b_i),$$

where n is an integer, $b_i - a_i \leqslant$ a positive constant c, and

$$i \neq j \Rightarrow [a_i, b_i) \cap [a_j, b_j) = \phi \text{(the empty set)}.$$

Consider the following two measures on R:

- the restricted Lebesgue measure β, so $\beta\{[a_i, b_i)\} = b_i - a_i$; and
- the measure μ defined by

  $$\mu\{[a_i, b_i)\} = F_s[\min\{b_i, L-1\}, \bar{L}] - F_s[\max\{a_i, 1-L\}, \bar{L}]$$

  for $F_s(\cdot, \bar{L})$ as in Section 4.3. Hence there exists a $k > 0$ such that $\mu\{[a_i, b_i)\} < k(b_i - a_i)$ for all $[a_i, b_i)$ of the form described above.

By the additivity of measures and the distributive law for R (under set union and intersection), $\mu \leqslant k\beta$ on all of R. Also, it is a standard result of measure theory that both measures can be extended uniquely to the class $S(R)$ of all Borel sets, which extensions we denote by $\tilde{\beta}$ and $\tilde{\mu}$.

For any $E \in S(R)$, there is clearly a collection of disjoint intervals A_i such that

$$E \subset \bigcup_{i=1}^{\infty} A_i$$

and

$$A_i \in R, \; i = 1, 2, \ldots .$$

Since μ is induced by a probability measure, it is finite, so

$$\tilde{u}[\bigcup_{i=1}^{\infty} A_i] = \sum_{i=1}^{\infty} \mu(A_i) \quad \text{(since the } A_i \text{ are disjoint)}$$

$$\leqslant \sum_{i=1}^{\infty} k\beta(A_i) \quad \text{(by assumption)}$$

$$= k \sum_{i=1}^{\infty} \beta(A_i) = k\tilde{\beta}[\bigcup_{i=1}^{\infty} A_i].$$

This must be true for the infimum of all such covers of E, so by the standard construction of the extensions $\tilde{\mu}$ and $\tilde{\beta}$,

$$\tilde{\mu} \leqslant k\tilde{\beta} \text{ on all of } S(R).$$

By the construction of the standard completions of $\tilde{\mu}$ and $\tilde{\beta}$ on the class $\overline{S(R)}$ of Lebesgue measureable sets, therefore,

$$\bar{\mu} \leqslant k\lambda \text{ on all of } \overline{S(R)},$$

where $\bar{\mu}$ denotes the completion of $\tilde{u}$ and λ denotes Lebesgue measure (the completion of $\bar{\beta}$). This means that $\bar{\mu}$ is absolutely continuous with respect to Lebesgue measure. Thus, by the Radon-Nikodym theorem, the definition of probability distribution functions, and the standard construction of μ and $\bar{\mu}$ from the quasi-distribution function $F_s(\cdot,\bar{L})$, the function $F_s(x,\bar{L})$ has a measurable Radon-Nikodym derivative with respect to Lebesgue measure on x for $1 - L < x < L - 1$. The same argument also holds with $F_s(x,L)$ in place of $F_s(x,\bar{L})$ everywhere.

Now, in the context of Eq. (22) and (24) of Section 4.3, let

$$\epsilon = \frac{1}{2}\Pr\{|t| \geqslant L\}$$

and let $f(\theta)$ denote some Radon-Nikodym derivative of $F_s(\theta,\bar{L})$. Hence, for all $\Delta > 0$ and all $\theta \in (1 - L, L - 1 - \Delta)$,

$$\Pr\{s \in [\theta,\ \theta + \Delta)\} = \int_\theta^{\theta+\Delta} f\, d\lambda < \epsilon\Delta.$$

Since f is measureable, we can define

$$E = \{\theta \in (1 - L, L - 1) \colon f(\theta) > 2\epsilon\}$$

and

$$m = \lambda(E).$$

From the construction of Lebesgue measure, for every $\delta > 0$ there exists a disjoint sequence of intervals $[a_i,\ b_i) = F_i$ such that

$$B = \bigcup_{i=1}^{\infty} [a_i, b_i) \supset E$$

and

$$\lambda(B) < m + \delta.$$

Therefore, integration with respect to Lebesgue measure gives

$$\int_B f = \int_E f + \int_{B-E} f \geqslant 2\epsilon m + \bar{\mu}(B - E) \geqslant 2\epsilon m$$

and

$$\int_B f = \sum_{i=1}^{\infty} \left[\int_{F_i} f\right] < \sum_{i=1}^{\infty} \epsilon(b_i - a_i) < \epsilon(m + \delta).$$

Hence, $\epsilon\delta > \epsilon m$ for all $\delta > 0$. Since $\epsilon > 0$, $m = \lambda(E) = 0$.

Now let $G_\delta = \{\theta \ \in \ (1 - L, L - 1) \colon f(\theta) < -\delta\}$ for any $\delta > 0$. Then

$$-\delta\lambda(G_\delta) \geqslant \int_{G_\delta} f d\lambda = \Pr\{s \in G_\delta \text{ and } |t| \geqslant L\} \geqslant 0,$$

so $\lambda(G_\delta) = 0$. Combining results, we get

$$\lambda\{\theta \in (1 - L,\ L - 1) \colon f(\theta) > 2\epsilon \text{ or } f(\theta) < -\delta\} = 0$$

for every $\delta > 0$. Finally,

$$\{\theta \in (1 - L,\ L - 1) \colon f(\theta) \in [0, 2\epsilon]\} = \bigcup_{h>1}^{\infty}\left\{\theta \in (1 - L,\ L - 1) \colon f(\theta) \notin \left[-\frac{1}{h}, 2\epsilon\right]\right\}$$

and

$$\lambda\{\theta \in (1-L,\ L-1): f(\theta) \notin [0, 2\epsilon]\} \leqslant \sum_{h>1}^{\infty} \lambda\left\{\theta \in (1-L,\ L-1): f(\theta) \notin \left[-\frac{1}{h},\ 2\epsilon\right]\right\} = 0.$$

Since Lebesgue measure is positive,

$$0 \leqslant f(\theta) \leqslant \epsilon = \Pr\{|t| \geqslant L\} \text{ for all } \theta \in (1-L,\ L-1),$$

except perhaps for a set A of Lebesgue measure zero, in which case another Radon-Nikodym derivative g can be constructed as

$$g(\theta) = \begin{cases} f(\theta) \text{ if } \theta \notin A \\ 0 \text{ if } \theta \in A, \end{cases}$$

which satisfies these inequalities for all $\theta \in (1-L,\ L-1)$.

Appendix E
SOME INEQUALITIES

For x, k, $a > 0$,

$$(x + a)^n \leqslant [\max\{a, kn\}^n e^{\min\{0, \frac{a}{k} - n\}}] e^{\frac{x}{k}};\ n = 1, 2, \ldots . \tag{E1}$$

This is established by noting that $(x + a)^n e^{-\frac{x}{k}}$ has the x derivative

$$(x + a)^{n-1} e^{-\frac{x}{k}} \left(n - \frac{x + a}{k} \right)$$

for all positive x. This derivative is positive if $kn > x + a$ and negative if $kn < x + a$. Hence the maximum value of $(x + a)^n e^{-\frac{x}{k}}$ for $x \geqslant 0$ occurs at $x = \max\{0, kn - a\}$. Substituting this value for x gives

$$(x + a)^n e^{-x/k} \leqslant \begin{cases} (kn)^n e^{\frac{a}{k} - n} & \text{if } kn \geqslant a \\ a^n & \text{if } kn \leqslant a, \end{cases}$$

which is equivalent to the desired inequality.

For t, $a \geqslant 0$,

$$(t + a)^n \leqslant 2^n (t^n + a^n), \tag{E2}$$

because

$$(t + a)^n \leqslant [2 \max\{t, a\}]^n \leqslant 2^n \max\{t^n, a^n\} \leqslant 2^n (t^n + a^n).$$